Problem 10.1.3

By Theorem 10.1, the mean of the output is

\[ \mu_Y = \mu_X \int_{-\infty}^{\infty} h(t) \, dt \]
\[ = 4 \int_{0}^{\infty} e^{-t/4a} \, dt \]
\[ = -4ae^{-t/4a} \bigg|_{0}^{\infty} \]
\[ = 4a \]

Since \( \mu_Y = 1 = 4a \), we must have \( a = 1/4 \).

Problem 10.2.4

(a) Note that \( |H(f)| = 1 \). This implies \( S_{\hat{M}}(f) = S_M(f) \). Thus the average power of \( \hat{M}(t) \) is

\[ \hat{q} = \int_{-\infty}^{\infty} S_{\hat{M}}(f) \, df = \int_{-\infty}^{\infty} S_M(f) \, df = q \]

(b) The average power of the upper sideband signal is

\[ E[U^2(t)] = E[M^2(t) \cos^2(2\pi f_c t + \Theta)] \]
\[ - E[2M(t)\hat{M}(t) \cos(2\pi f_c t + \Theta) \sin(2\pi f_c t + \Theta)] \]
\[ + E[\hat{M}^2(t) \sin^2(2\pi f_c t + \Theta)] \]

To find the expected value of the random phase cosine, for an integer \( n \neq 0 \), we evaluate

\[ E[\cos(2\pi f_c t + n\Theta)] = \int_{-\infty}^{\infty} \cos(2\pi f_c t + n\theta) f_{\theta}(\theta) \, d\theta \]
\[ = \int_{0}^{2\pi} \cos(2\pi f_c t + n\theta) \frac{1}{2\pi} d\theta \]
\[ = \frac{1}{2\pi} \sin(2\pi f_c t + n\theta) \bigg|_{0}^{2\pi} \]
\[ = \frac{1}{2\pi} (\sin(2\pi f_c t + 2\pi n) - \sin(2\pi f_c t)) \]
\[ = 0 \]

Similar steps will show that for any integer \( n \neq 0 \), the random phase sine also has expected value

\[ E[\sin(2\pi f_c t + n\Theta)] = 0 \]
Using the trigonometric identity \( \cos^2 \phi = (1 + \cos 2\phi)/2 \), we can show

\[
E[\cos^2(2\pi f_c t + \Theta)] = E\left[ \frac{1}{2} (1 + \cos(2\pi(2f_c)t + 2\Theta)) \right] = 1/2
\]

Similarly,

\[
E[\sin^2(2\pi f_c t + \Theta)] = E\left[ \frac{1}{2} (1 - \cos(2\pi(2f_c)t + 2\Theta)) \right] = 1/2
\]

In addition, the identity \( 2 \sin \phi \cos \phi = \sin 2\phi \) implies

\[
E[2\sin(2\pi f_c t + \Theta) \cos(2\pi f_c t + \Theta)] = E[\cos(4\pi f_c t + 2\Theta)] = 0
\]

Since \( M(t) \) and \( \hat{M}(t) \) are independent of \( \Theta \), the average power of the upper sideband signal is

\[
E[U^2(t)] = E[M^2(t)] E[\cos^2(2\pi f_c t + \Theta)] + E[\hat{M}^2(t)] E[\sin^2(2\pi f_c t + \Theta)]
\]

\[
- E[M(t)\hat{M}(t)] E[2\cos(2\pi f_c t + \Theta) \sin(2\pi f_c t + \Theta)]
\]

\[
= q/2 + q/2 + 0 = q
\]

**Problem 10.3.3**

Theorem 10.9 which states

\[
S_{XY}(f) = H(f)S_X(f)
\]

(a) From Table 10.1, we observe that

\[
S_X(f) = \frac{8}{16 + (2\pi f)^2}, \quad H(f) = \frac{1}{7 + j2\pi f}
\]

(b) From Theorem 10.9,

\[
S_{XY}(f) = H(f)S_X(f) = \frac{8}{[7 + j2\pi f][16 + (2\pi f)^2]}
\]

(c) To find the cross correlation, we need to find the inverse Fourier transform of \( S_{XY}(f) \). A straightforward way to do this is to use a partial fraction expansion of \( S_{XY}(f) \). That is, by defining \( s = j2\pi f \), we observe that

\[
\frac{8}{(7 + s)(4 + s)(4 - s)} = \frac{-8/33}{7 + s} + \frac{1/3}{4 + s} + \frac{1/11}{4 - s}
\]

Hence, we can write the cross spectral density as

\[
S_{XY}(f) = \frac{-8/33}{7 + j2\pi f} + \frac{1/3}{4 + j2\pi f} + \frac{1/11}{4 - j\pi f}
\]
Unfortunately, terms like $1/(a - j2\pi f)$ do not have an inverse transforms. The solution is to write $S_{XY}(f)$ in the following way:

$$S_{XY}(f) = \frac{-8/33}{7+j2\pi f} + \frac{8/33}{4+j2\pi f} + \frac{1/11}{4+j2\pi f} + \frac{1/11}{4-j2\pi f}$$

$$= \frac{-8/33}{7+j2\pi f} + \frac{8/33}{4+j2\pi f} + \frac{1}{8/11} + \frac{1}{16+(2\pi f)^2}$$

Now, we see from Table 10.1 that the inverse transform is

$$R_{XY}(\tau) = -\frac{8}{33}e^{-7\tau}u(\tau) + \frac{8}{33}e^{-4\tau}u(\tau) + \frac{1}{11}e^{-4|\tau|}$$

### Problem 10.4.1

(a) Since $C_X(t_1, t_2 - t_1) = \rho \sigma_1 \sigma_2$, the covariance matrix is

$$C = \begin{bmatrix} C_X(t_1, 0) & C_X(t_1, t_2 - t_1) \\ C_X(t_2, t_1 - t_2) & C_X(t_2, 0) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$$

Since $C$ is a $2 \times 2$ matrix, it has determinant $|C| = \sigma_1^2 \sigma_2^2 (1 - \rho^2)$.

(b) Is is easy to verify that

$$C^{-1} = \frac{1}{1 - \rho^2} \begin{bmatrix} \frac{1}{\sigma_1^2} & -\rho \\ -\frac{\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

(c) The general form of the multivariate density for $X(t_1), X(t_2)$ is

$$f_{X(t_1), X(t_2)}(x_1, x_2) = \frac{1}{(2\pi)^{k/2} |C|^{1/2}} e^{-\frac{1}{2}(x-\bar{\mu}_X)^T C^{-1} (x-\bar{\mu}_X)}$$

where $k = 2$ and $\bar{x} = [x_1, x_2]^T$ and $\bar{\mu}_X = [\mu_1, \mu_2]^T$. Hence,

$$\frac{1}{(2\pi)^{k/2} |C|^{1/2}} = \frac{1}{2\pi \sigma_1^2 \sigma_2^2 (1 - \rho^2)}$$

Furthermore, the exponent is

$$-\frac{1}{2}(x-\bar{\mu}_X)^T C^{-1} (x-\bar{\mu}_X) = -\frac{1}{2} [x_1 - \mu_1, x_2 - \mu_2] \frac{1}{1 - \rho^2} \begin{bmatrix} \frac{1}{\sigma_1^2} & -\rho \\ -\frac{\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= -\left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - \frac{2\rho(x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_1 \sigma_2} + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2$$

Plugging in each piece into the joint PDF $f_{X(t_1), X(t_2)}(x_1, x_2)$ given above, we obtain the bivariate Gaussian PDF.
Problem 10.5.3

(a) Since $S_W(f) = 10^{-15}$ for all $f$, $R_W(\tau) = 10^{-15} \delta(\tau)$.

(b) Since $\Theta$ is independent of $W(t)$,

$$E[V(t)] = E[W(t) \cos(2\pi f_c t + \Theta)] = E[W(t)] E[\cos(2\pi f_c t + \Theta)] = 0$$

(c) We cannot initially assume $V(t)$ is WSS so we first find

$$R_V(t, \tau) = E[V(t)V(t + \tau)] = E[W(t)W(t + \tau) \cos(2\pi f_c (t + \tau) + \Theta)] = 10^{-15} \delta(\tau)E[\cos(2\pi f_c t + \Theta) \cos(2\pi f_c(t + \tau) + \Theta)]$$

We see that for all $\tau \neq 0$, $R_V(t, t + \tau) = 0$. Thus we need to find the expected value of

$$E[\cos(2\pi f_c t + \Theta) \cos(2\pi f_c(t + \tau) + \Theta)]$$

only at $\tau = 0$. However, its good practice to solve for arbitrary $\tau$:

$$E[\cos(2\pi f_c t + \Theta) \cos(2\pi f_c(t + \tau) + \Theta)]$$

$$= \frac{1}{2}E[\cos(2\pi f_c t) + \cos(2\pi f_c(2t + \tau) + 2\Theta)]$$

$$= \frac{1}{2} \cos(2\pi f_c t) + \frac{1}{2} \int_{0}^{2\pi} \cos(2\pi f_c(2t + \tau) + 2\Theta) \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2} \cos(2\pi f_c t) + \frac{1}{2} \sin(2\pi f_c(2t + \tau) + 2\theta) \bigg|_{0}^{2\pi}$$

$$= \frac{1}{2} \cos(2\pi f_c t) + \frac{1}{2} \sin(2\pi f_c(2t + \tau) + 4\pi) - \frac{1}{2} \sin(2\pi f_c(2t + \tau))$$

$$= \frac{1}{2} \cos(2\pi f_c t)$$

Consequently,

$$R_V(t, \tau) = \frac{1}{2} 10^{-15} \delta(\tau) \cos(2\pi f_c t) = \frac{1}{2} 10^{-15} \delta(\tau)$$

(d) Since $E[V(t)] = 0$ and since $R_V(t, \tau) = R_V(\tau)$, we see that $V(t)$ is a wide sense stationary process. Since $L(f)$ is a linear time invariant filter, the filter output $Y(t)$ is also a wide sense stationary process.

(e) The filter input $V(t)$ has power spectral density $S_V(f) = \frac{1}{2} 10^{-15}$. The filter output has power spectral density

$$S_Y(f) = |L(f)|^2 S_V(f) = \begin{cases} 10^{-15}/2 & |f| \leq B \\ 0 & \text{otherwise} \end{cases}$$

The average power of $Y(t)$ is

$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f) df = \int_{-B}^{B} \frac{1}{2} 10^{-15} df = 10^{-15} B$$
Problem 10.5.4
and $Y(t)$ are the input and output of a linear time invariant filter $h(u)$. In that case,
\[
Y(t) = \int_{0}^{t} N(u) \, du = \int_{-\infty}^{\infty} h(t-u)N(u) \, du
\]
For the above two integrals to be the same, we must have
\[
h(t-u) = \begin{cases} 
1 & 0 \leq t-u \leq t \\
0 & \text{otherwise}
\end{cases}
\]
Making the substitution $v = t-u$, we have
\[
h(v) = \begin{cases} 
1 & 0 \leq v \leq t \\
0 & \text{otherwise}
\end{cases}
\]
Thus the impulse response $h(v)$ depends on $t$. That is, the filter response is linear but not time invariant. Since Theorem 10.7 requires that $h(t)$ be time invariant, this example does not violate the theorem.

Problem 10.5.5
process since it is the output of a linear filter with Gaussian input process $N(t)$. We observe that $E[Y(t)] = \int_{0}^{t} E[N(u)] \, du = 0$. The autocorrelation function of the output is
\[
R_{Y}(t, \tau) = E[Y(t)Y(t+\tau)]
\]
\[
= E \left[ \left( \int_{0}^{t} N(u) \, du \right) \left( \int_{0}^{t+\tau} N(v) \, dv \right) \right]
\]
\[
= \int_{0}^{t} \int_{0}^{t+\tau} E[N(u)N(v)] \, dv \, du
\]
Since $N(t)$ is a White noise process,
\[
E[N(u)N(v)] = R_{N}(u, v-u) = \alpha \delta(v-u)
\]
This implies
\[
R_{Y}(t, \tau) = \alpha \int_{0}^{t} \left( \int_{0}^{t+\tau} \delta(v-u) \, dv \right) \, du
\]
If $\tau \geq 0$, then for each $u \in [0, t]$ there is $v \in [0, t+\tau]$ such that $v = u$ and $\int_{0}^{t+\tau} \delta(v-u) \, dv = 1$. This implies
\[
R_{Y}(t, \tau) = \alpha \int_{0}^{t} du = \alpha t
\]
If $\tau < 0$, then we write
\[
R_{Y}(t, \tau) = \alpha \int_{0}^{t+\tau} \left( \int_{0}^{t} \delta(v-u) \, du \right) \, dv
\]

5
For each \( v \in [0, t + \tau] \) there is \( u \in [0, t] \) such that \( u = v \) and \( \int_0^t \delta(v-u)\,du = 1 \). This implies
\[
R_Y(t, \tau) = \alpha \int_0^{t+\tau} \,dv = \alpha(t + \tau)
\]
The complete expression for the autocorrelation of \( Y(t) \) is
\[
R_Y(t, \tau) = \alpha \min(t, t + \tau)
\]
In Chapter 6, we found that a Brownian motion process \( X(t) \) is a zero mean Gaussian process. In addition, in Example 6.17, we found that a Brownian motion process \( X(t) \) has autocorrelation function
\[
R_X(t, \tau) = \alpha \min(t, t + \tau)
\]
Since a Gaussian process is completely specified by the mean \( E[X(t)] \) and the autocorrelation \( R_X(t, \tau) \), we can conclude that \( Y(t) \) must be a Gaussian process.

Another way to interpret this result is to write for \( t > s \), the increment is
\[
Y(t) - Y(s) = \int_0^t N(u)\,du - \int_0^s N(u)\,du = \int_s^t N(u)\,du
\]
For each \( v \in [s,t] \), \( N(v) \) is independent of \( N(u) \) for any \( u \in [0,s] \). Thus for any \( s' \leq s \), \( Y(s') = \int_0^{s'} N(u)\,du \) is independent of \( Y(t) - Y(s) \). Hence \( Y(t) \) is a zero mean Gaussian process with independent increments and \( Y(0) = 0 \), which is Definition 6.11 for the Brownian motion process.

**Problem 10.6.4**

We start with Theorem 10.13:
\[
R_Y[n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j R_X[n+i-j]
= R_X[n-1] + 2R_X[n] + R_X[n+1]
\]
First we observe that for \( n \leq -2 \) or \( n \geq 2 \),
\[
R_Y[n] = R_X[n-1] + 2R_X[n] + R_X[n+1] = 0
\]
This suggests that \( R_X[n] = 0 \) for \( |n| > 1 \). In addition, we have the following facts:
\[
R_Y[0] = R_X[-1] + 2R_X[0] + R_X[1] = 2
R_Y[-1] = R_X[-2] + 2R_X[-1] + R_X[0] = 1
\]
A simple solution to this set of equations is \( R_X[0] = 1 \) and \( R_X[n] = 0 \) for \( n \neq 0 \).