Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Roy D. Yates and David J. Goodman

Problem Solutions : Yates and Goodman,1.2.3 1.3.1 1.4.6 1.4.7 1.4.8 1.5.6 1.6.7 1.6.9 1.7.5 1.7.7 1.8.7 and 1.9.9

Problem 1.2.3

The sample space is

$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\}$$

The event H is the set

$$H = \{A\heartsuit, \ldots, K\heartsuit\}$$

Problem 1.3.1

The sample space of the experiment is

$$S = \{LF, BF, LW, BW\}$$

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1 - 0.5 - 0.2 - 0.2 = 0.1$. The questions can be answered using Theorem 1.5.

(a) The probability that a program is slow is

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3.$$

(b) The probability that a program is big is

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4.$$

(c) The probability that a program is slow or big is

$$P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5$$

Problem 1.4.6

It is tempting to use the following proof:

> Since S and ϕ are mutually exclusive, and since $S = S \cup \phi$,
>
> $$1 = P[S \cup \phi] = P[S] + P[\phi]$$
>
> Since $P[S] = 1$, we must have $P[\phi] = 0$.

The above "proof" used the property that for mutually exclusive sets A_1 and A_2,

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]$$

The problem is that this property is a consequence of the three axioms, and thus must be proven. For a proof that uses just the three axioms, let A_1 be an arbitrary set and for $n = 2, 3, \ldots$, let $A_n = \phi$. Since $A_1 = \cup_{i=1}^{\infty} A_i$, we can use Axiom 3 to write

$$P[A_1] = P[\cup_{i=1}^{\infty} A_i] = P[A_1] + P[A_2] + \sum_{i=3}^{\infty} P[A_i]$$

By subtracting $P[A_1]$ from both sides, the fact that $A_2 = \phi$ permits us to write

$$P[\phi] + \sum_{n=3}^{\infty} P[A_i] = 0$$

By Axiom 1, $P[A_i] \geq 0$ for all i. Thus, $\sum_{n=3}^{\infty} P[A_i] \geq 0$. This implies $P[\phi] \leq 0$. Since Axiom 1 requires $P[\phi] \geq 0$, we must have $P[\phi] = 0$.

Problem 1.4.7

Following the hint, we define the set of events $\{A_i | i = 1, 2, \ldots\}$ such that $i = 1, \ldots, m$, $A_i = B_i$ and for $i > m$, $A_i = \phi$. By construction, $\cup_{i=1}^{m} B_i = \cup_{i=1}^{\infty} A_i$. Axiom 3 then implies

$$P[\cup_{i=1}^{m} B_i] = P[\cup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} P[A_i]$$

For $i > m$, $P[A_i] = 0$, yielding

$$P[\cup_{i=1}^{m} B_i] = \sum_{i=1}^{m} P[A_i] = \sum_{i=1}^{m} P[B_i]$$

Problem 1.4.8

Theorem 1.7 requires a proof from which we can check which axioms are used. However, the problem is somewhat hard because there may still be a simpler proof that uses fewer axioms. Still, the proof of each part will need Theorem 1.4 which we now prove.

For the mutually exclusive events $B_1, \ldots, B_m$, let $A_i = B_i$ for $i = 1, \ldots, m$ and let $A_i = \phi$ for $i > m$. In that case, by Axiom 3,

$$\begin{aligned} P[B_1 \cup B_2 \cup \cdots \cup B_m] &= P[A_1 \cup A_2 \cup \cdots] \\ &= \sum_{i=1}^{m-1} P[A_i] + \sum_{i=m}^{\infty} P[A_i] \\ &= \sum_{i=1}^{m-1} P[B_i] + \sum_{i=m}^{\infty} P[A_i] \end{aligned}$$

Now, we use Axiom 3 again on $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^{\infty} P[A_i] = P[A_m \cup A_{m+1} \cup \cdots] = P[B_m]$$

Thus, we have used just Axiom 3 to prove Theorem 1.4:

$$P[B_1 \cup B_2 \cup \cdots \cup B_m] = \sum_{i=1}^{m} P[B_i]$$

(a) To show $P[\phi] = 0$, let $B_1 = S$ and let $B_2 = \phi$. Thus by Theorem 1.4

$$P[S] = P[B_1 \cup B_2] = P[B_1] + P[B_2] = P[S] + P[\phi]$$

Thus, $P[\phi] = 0$. Note that this proof uses only Theorem 1.4 which uses only Axiom 3.

(b) Using Theorem 1.4 with $B_1 = A$ and $B_2 = A^c$, we have

$$P[S] = P[A \cup A^c] = P[A] + P[A^c]$$

Since, Axiom 2 says $P[S] = 1$, $P[A^c] = 1 - P[A]$. This proof uses Axioms 2 and 3.

(c) By Theorem 1.2, we can write both A and B as unions of disjoint events:

$$A = (AB) \cup (AB^c) \qquad B = (AB) \cup (A^cB)$$

Now we apply Theorem 1.4 to write

$$P[A] = P[AB] + P[AB^c] \qquad P[B] = P[AB] + P[A^cB]$$

We can rewrite these facts as

$$P[AB^c] = P[A] - P[AB] \qquad P[A^cB] = P[B] - P[AB] \tag{1}$$

Note that so far we have used only Axiom 3. Finally, we observe that $A \cup B$ can be written as the union of mutually exclusive events

$$A \cup B = (AB) \cup (AB^c) \cup (A^cB)$$

Once again, using Theorem 1.4, we have

$$P[A \cup B] = P[AB] + P[AB^c] + P[A^cB] \tag{2}$$

Substituting the results of Equation 1 into Equation 2 yields

$$P[A \cup B] = P[AB] + P[A] - P[AB] + P[B] - P[AB]$$

which completes the proof. Note that this claim required only Axiom 3.

(d) Observe that since $A \subset B$, we can write B as the disjoint union $B = A \cup (A^cB)$. By Theorem 1.4 (which uses Axiom 3),

$$P[B] = P[A] + P[A^cB]$$

By Axiom 1, $P[A^cB] \geq 0$, hich implies $P[A] \leq P[B]$. This proof uses Axioms 1 and 3.

Problem 1.5.6
The problem statement yields the obvious facts that $P[L] = 0.16$ and $P[H] = 0.10$. The words "10% of the ticks that had either Lyme disease or HGE carried both diseases" can be written as

$$P[LH|L \cup H] = 0.10$$

(a) Since $LH \subset L \cup H$,

$$P[LH|L \cup H] = \frac{P[LH \cap (L \cup H)]}{P[L \cup H]} = \frac{P[LH]}{P[L \cup H]} = 0.10$$

Thus,

$$P[LH] = 0.10P[L \cup H] = 0.10\,(P[L] + P[H] - P[LH])$$

Since $P[L] = 0.16$ and $P[H] = 0.10$,

$$P[LH] = \frac{0.10\,(0.16 + 0.10)}{1.1} = 0.0236$$

(b) The conditional probability that a tick has Lyme disease given that it has HGE is

$$P[L|H] = \frac{P[LH]}{P[H]} = 0.236$$

Problem 1.6.7

(a) For any events A and B, we can write the law of total probability in the form of

$$P[A] = P[AB] + P[AB^c]$$

Since A and B are independent, $P[AB] = P[A]P[B]$. This implies

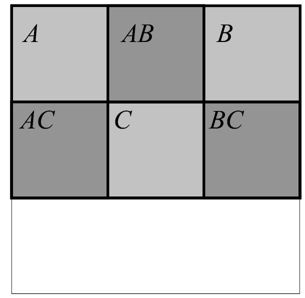
$$P[AB^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]$$

Thus A and B^c are independent.

(b) Proving that A^c and B are independent is not really necessary. Since A and B are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of A and B proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels A and B reversed.

(c) To prove that A^c and B^c are independent, we apply the result of part (a) to the sets A and B^c. Since we know from part (a) that A and B^c are independent, part (b) says that A^c and B^c are independent.

Problem 1.6.9



In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, A, B, and C each have area $1/3$ and thus probability $1/3$. The three way intersection ABC has zero probability, implying A, B, and C are not mutually independent since

$$P[ABC] = 0 \neq P[A]P[B]P[C]$$

However, AB, BC, and AC each has area $1/9$. As a result, each pair of events is independent since

$$P[AB] = P[A]P[B] \qquad P[BC] = P[B]P[C] \qquad P[AC] = P[A]P[C]$$

Problem 1.7.5

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[+,H]}{P[+]} = \frac{P[+,H]}{P[+,H] + P[+,H^c]}$$

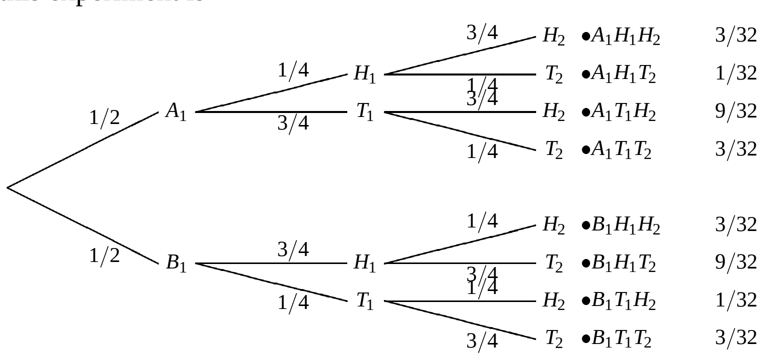
We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} P[H|+] &= \frac{P[+|H]P[H]}{P[+|H]P[H] + P[+|H^c]P[H^c]} \\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \\ &= 0.0194 \end{aligned}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 1.7.7

The tree for this experiment is



The event H_1H_2 that heads occurs on both flips has probability

$$P[H_1H_2] = P[A_1H_1H_2] + P[B_1H_1H_2] = 6/32$$

The probability of H_1 is

$$P[H_1] = P[A_1H_1H_2] + P[A_1H_1T_2] + P[B_1H_1H_2] + P[B_1H_1T_2] = 1/2$$

Similarly,

$$P[H_2] = P[A_1H_1H_2] + P[A_1T_1H_2] + P[B_1H_1H_2] + P[B_1T_1H_2] = 1/2$$

Thus $P[H_1H_2] \neq P[H_1]P[H_2]$, implying H_1 and H_2 are not independent. This result should not be surprising since if the first flip is heads, it is likely that coin B was picked first. In this case, the second flip is less likely to be heads since it becomes more likely that the second coin flipped was coin A.

Problem 1.8.7

What our design must specify is the number of boxes on the ticket, and the number of specially marked boxes. Suppose each ticket has n boxes and $5+k$ specially marked boxes. Note that when $k > 0$, a winning ticket will still have k unscratched boxes with the special mark. A ticket is a winner if each time a box is scratched off, the box has the special mark. Assuming the boxes are scratched off randomly, the first box scratched off has the mark with probability $(5+k)/n$ since there are $5+k$ marked boxes out of n boxes. Moreover, if the first scratched box has the mark, then there are $4+k$ marked boxes out of $n-1$ remaining boxes. Continuing this argument, the probability that a ticket is a winner is

$$p = \frac{5+k}{n}\frac{4+k}{n-1}\frac{3+k}{n-2}\frac{2+k}{n-3}\frac{1+k}{n-4} = \frac{(k+5)!(n-5)!}{k!n!}$$

By careful choice of n and k, we can choose p close to 0.01. For example, some possible choices are

n	k	p
9	0	0.0079
11	1	0.012
14	2	0.0105
17	3	0.0090

Probably, a gamecard with $N = 14$ boxes and $5+k = 7$ shaded boxes would be quite reasonable.

Problem 1.9.9

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3 \qquad P[G_2] = 2/3$$

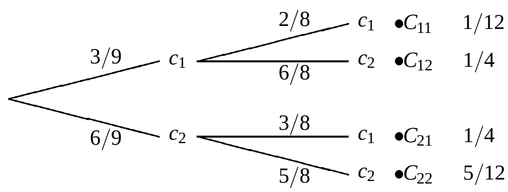
In addition, the problem statement tells us that

$$P[K|G_1] = 1/2 \qquad P[K|G_2] = 1/3$$

Combining these facts using the Law of Total Probability yields

$$P[K] = P[K|G_1]P[G_1] + P[K|G_2]P[G_2] = (1/2)(1/3) + (1/3)(2/3) = 7/18$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and he second kicker from group j. The experiment to choose the kickers is described by the sample tree:



Since a kicker from group 1 makes a kick with probability $1/2$ while a kicker from group 2 makes a kick with probability $1/3$,

$$P[K_1K_2|C_{11}] = (1/2)^2 \qquad P[K_1K_2|C_{12}] = (1/2)(1/3)$$
$$P[K_1K_2|C_{21}] = (1/3)(1/2) \qquad P[K_1K_2|C_{22}] = (1/3)^2$$

By the law of total probability,

$$\begin{aligned} P[K_1K_2] &= P[K_1K_2|C_{11}]P[C_{11}] + P[K_1K_2|C_{12}]P[C_{12}] \\ &\quad + P[K_1K_2|C_{21}]P[C_{21}] + P[K_1K_2|C_{22}]P[C_{22}] \\ &= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} \\ &= 15/96 \end{aligned}$$

It should be apparent that $P[K_1] = P[K]$ from part (a). Symmetry should also make it clear that $P[K_1] = P[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $P\left[K_2|C_{ij}\right]$ and using the law of total probability to calculate $P[K_2]$.

$$P[K_2|C_{11}] = 1/2 \qquad P[K_2|C_{12}] = 1/3$$
$$P[K_2|C_{21}] = 1/2 \qquad P[K_2|C_{22}] = 1/3$$

By the law of total probability,

$$\begin{aligned} P[K_2] &= P[K_2|C_{11}]P[C_{11}] + P[K_2|C_{12}]P[C_{12}] + P[K_2|C_{21}]P[C_{21}] + P[K_2|C_{22}]P[C_{22}] \\ &= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} \\ &= 7/18 \end{aligned}$$

We observe that K_1 and K_2 are not independent since

$$P[K_1K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = P[K_1]P[K_2]$$

Note that $15/96$ and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss.

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is $1/2$. If the kicker is from group 2, the success probability is $1/3$. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

$$P[M|G_1] = \binom{10}{5}(1/2)^5(1/2)^5 \qquad P[M|G_2] = \binom{10}{5}(1/3)^5(2/3)^5$$

We use the Law of Total Probability to find

$$\begin{aligned} P[M] &= P[M|G_1]P[G_1] + P[M|G_2]P[G_2] \\ &= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right) \end{aligned}$$