THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:426 **Wireless PCS Problems** **Spring 2006**

1. **Ricean Distribution:** We talked about the Ricean distribution which arises when you have a strong line of sight propagation component coupled to random but much smaller signals arriving with random amplitudes and phases at a receiver.

   Well, derive it. Use the Rayleigh workup as a base, but add in a strong line of sight (say, in-phase) component. You may use any resources you like (except each other) for this problem.

   Solution: *We start by modeling the incoming signal as*

$$x(t) = A\cos 2\pi f_c t + \sum_k b_k \cos(2\pi f_c t - \phi_k)$$

where A is a known constant, the b_k are i.i.d. random variables and ϕ_k are also i.i.d. random variables, assumed uniformly distributed over $(0, 2\pi)$. In phasor notation we can write the signal envelope as

$$Z = (A+B) + Cj$$

where B and C are composed of sums of random variables. Specifically, $B = \sum_k b_k \cos\phi_k$ and $C = -\sum_k b_k \sin\phi_k$. We therefore approximate B and C as Gaussian. They are zero mean since the expectation with respect to the ϕ_k is zero. Further, we can also see that B and C are uncorrelated since $E[\cos\phi_k \sin\phi_\ell] = 0$ for uniform $\{\phi_k\}$.

So we take the leap of faith and assume B and C are independent Gaussians. They both have zero mean, and they both have the same variance which we'll call σ^2. Since A is added directly to B we can define a new random variable $B' = A+B$ which is still Gaussian, but with mean A.

NOTE: some students started from an in-phase/quadrature noise description, but did not properly move from the stochastic process to the above description and were docked points. A stochastic PROCESS is not a random variable and you can't treat it in the same way.

In any event, we now want to know the distribution on $|Z|$. Well

$$|Z| = (B')^2 + C^2$$

The probability that $|Z| < r$ is

$$Prob(|Z| < r) = \int_{x^2+y^2<R} \frac{1}{2\pi\sigma^2} e^{((x-A)^2+y^2)/2\sigma^2} dxdy$$

Using polar coordinates we have $x = R\cos\theta$ and the integral becomes

$$Prob(|Z| < r) = \int_0^{2\pi} \int_0^r \frac{1}{2\pi\sigma^2} e^{-(R^2+A^2-2AR\cos\theta)/2\sigma^2} dRd\theta$$

To find the PDF we differentiate with respect to r to obtain

$$f_{|Z|}(r) = \frac{1}{2\pi\sigma^2} e^{-(r^2+A^2)/2\sigma^2} \int_0^{2\pi} e^{Ar\cos\theta/\sigma^2} d\theta$$

which is the Ricean distribution. It turns out that the integral is a bessel function ($2\pi I_0(r)$), but that's just a name.

2. **Cora, Data the Dog and Marty the Angry Squirrel:**

Cora the Communications Engineer and her mostly deaf dog Data are out playing in a field one day. Data is doing what most dogs do in big fields. Cora is testing out her new wonder-goodie audio equipment – four loudspeakers, a special amplifier and a microphone – when she sees Martin T. Sciuris the squirrel, her arch nemesis. Data has a taste for squirrels, but Martin is both wiley and deadly, so Cora does not want Data to see Martin. Martin on the other hand, starts squealing delightedly trying to catch Data's attention.

Luckily, unbeknownst to Martin, Data is mostly deaf and ONLY hears sound at a particular frequency – 1000 Hz. However, from previous experience, Cora knows Martin's ears are highly sensitive, so if she whistles to Data on his frequency, Martin will hear and whistle himself, drawing Data to him and certain doom.

Cora thinks quickly and realizes she has four loudspeakers and her special home-brewed amplifier has the capacity to delay the signal going into each speaker by an arbitrary amount. What she'd like to do is set up the speakers in some spatial pattern around her, adjust the delays appropriately and then whistle into her microphone. From her knowledge of phased array antennas, she knows that if she phases things properly, Data will hear her whistle and come running, while Martin will hear almost nothing.

Assume Cora arranges her loudspeakers as in FIGURE 2, the speaker separations are six inches, Data makes an angle of θ with horizontal axis as shown while Martin makes an angle ω as shown. Also assume that the amplitudes of the sinusoids that Cora feeds into the loudspeakers are identical and that both Martin and Data are far away from the loudspeakers (say 1000 feet). You may treat the loudspeakers as isotropic point sources of sound and you may assume that sound travels at a speed of 1000 feet per second in air.

Please answer the following questions and carefully label any diagrams you supply. HINT: Carefully define your performance metric. Signal power ratios are often a useful measure.

(a) What delays d_i should Cora supply to each speaker so that the signal reaches Data with maximum intensity? Note that we do not care about Martin just yet. Sketch the radiation pattern which results from the resultant loudspeaker phasings (write a computer or Matlab program to do this).

Solution: *For the signal to reach Data with maximum intensity, we must ensure that the 4 sinusoids add up constructively at the dog's ear. In otherwords,the 4 sinusoids must be exactly in phase or out of phase by some multiple of 2π. The path difference between two speakers spaced 'd' apart is $d\sin(\theta)$. In terms of phase, this corresponds to an angle of*

$$\phi = (2\pi)\frac{d\sin\theta}{\lambda} \tag{1}$$

where λ is the wavelength of the sinusoid. In our case, the wavelength is given by $\lambda = v/f$ where v is the velocity of sound and f is the frequency. This gives us a wavelength of 1 foot, which is twice the spacing between adjacent speakers. Hence $\lambda = 2d$.

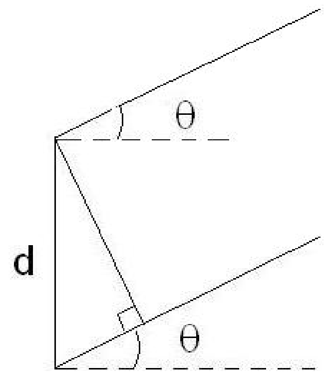

Figure 1: This shows how the path difference is calculated. Since the two (approximately) parallel lines stop at the same point, the difference between their path lengths is simply $dsin\theta$

Thus, the phase difference between two adjacent speakers is given by

$$\phi = (2\pi)\frac{dsin\theta}{2d} = \pi sin\theta \tag{2}$$

To make the adjacent speakers be exactly in phase, we delay the speaker with the shorter path length by t_0 which will contribute a phase angle to cancel the phase angle due to the path difference. The phase angle contributed will be simply $2000\pi t_0$, where t_0 must take a value so that this exceeds the phase angle due to the path difference by some integral multiple of 2π. Thus we must have

$$2000\pi t_0 = \pi sin\theta + 2n\pi \tag{3}$$

which implies that $t_0 = sin\theta/2000 + n/1000$ for any integer n.

This is the delay that must be between any two adjacent speakers. The furthest speaker from the dog is the lowermost speaker (spk-1). Let this have no delay. Then the speaker above it (spk-2) must have a delay of t_0. Similarly, spk-3 must lag spk-2 by t_0 and so it must have a total delay of $2t_0$ and lastly, spk-4 must have a delay of $3t_0$, where $t_0 = sin\theta/2000 + n/1000$. Note that since n can take any integral value, there are many possible delay values that will cause the waves to add up constructively at the dog.

We plot the radiation pattern for $\theta = 60$ degrees in Figure 2. Note the maximum power at this value of θ.

(b) Now assume the same set up as FIGURE 2, but now Cora needs to phase the loudspeakers so that Martin hears as close to nothing as possible. What delays d_i should Cora supply to each speaker? Sketch the radiation pattern which results from the resultant loudspeaker phasings (write a computer or Matlab program to do this).

Solution: *For martin to hear nothing, the waves must add up destructively at that point. In other words, they must be out of phase by an odd multiple of π. Following a similar line of reasoning as above, we get*

$$2000\pi t_0 = \pi sin\omega + 2(n+1)\pi \tag{4}$$

which gives us $t_0 = sin\omega/2000 + (2n+1)/2000$.

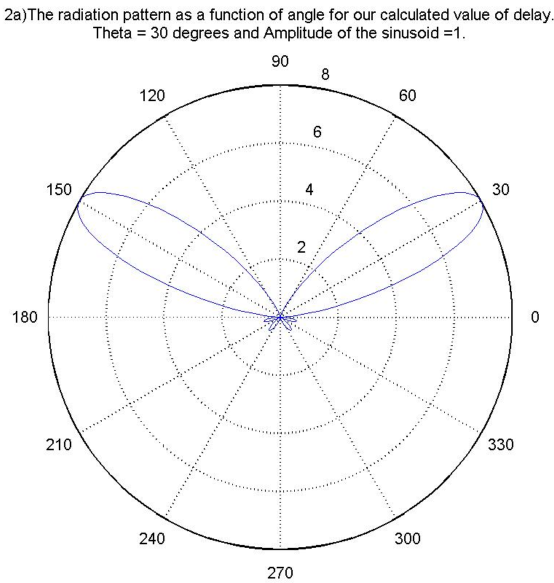

Figure 2: This shows the radiation pattern for 2a, with $\theta = 60$ degrees.

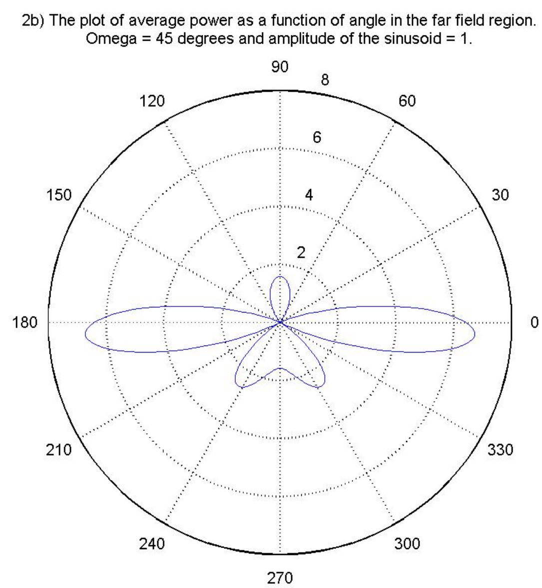

Figure 3: This shows the radiation pattern for 2b, with $\omega = 45$ degrees.

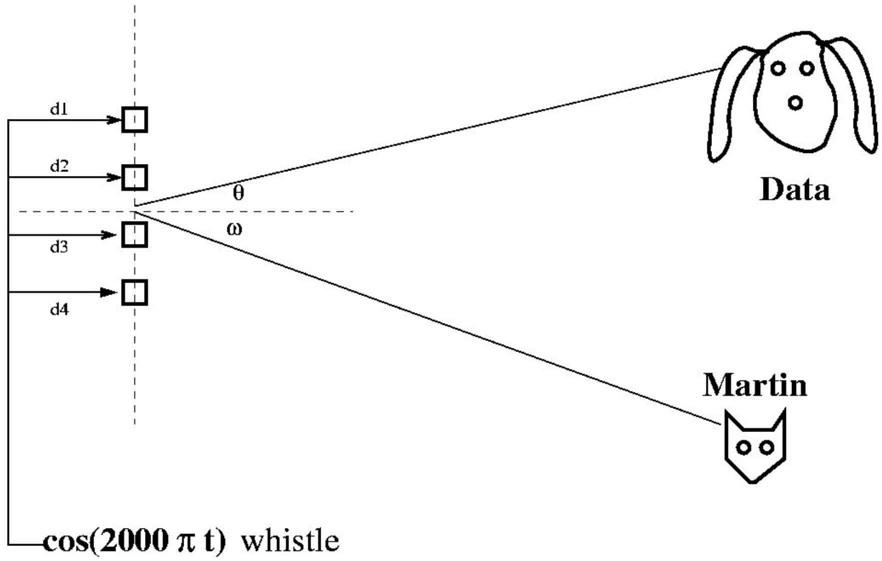

Figure 4: Figure for Cora, Data and Martin problem.

There are many possible solutions. In this solution I've chosen delays such that spk-1 and spk-2 cancel out, and spk-3 and spk-4 cancel out. Thus, our delays can be given by 0, t_0*,0 and* t_0 *(going from the uppermost to the lowermost) where* t_0 *is given this time by* $t_0 = sin\omega/2000 + (2n+1)/2000$.

We plot the graph for the radiation pattern for $\omega = 45$ *degrees in Figure 3. The graph shows us the null at* $\omega = 45$ *degrees. But note that even positions not too far from that value of* ω *are affected by the null and have very low powers. In other words, we don't get a sharp null. But this is the best we can do with just 4 antennas. With more antennas, we'd be able to restrict that null to a smaller angle range.*

(c) Now assume that Cora decides to be aggressive and instead of warning Data wants to send a strong 1KHz signal to blast Martin's ears off while sparing Data's ears. What should the d_i be?

Solution: *One way to do this is to simply calculate delays to put a null at the dog and then blast a sinusoid at very high power. In this way we're guaranteed that no power reaches the dog (since it is at a null). Then if* $\omega \neq \theta$ *we know that the squirrel does not lie at a null. Hence, we can blast power and know that some non-zero fraction of it will reach the squirrel.*

However, there are many ways to get a null at the dog. In order to do the best, we must choose the delay combination that maximizes the intensity at Martin while ensuring that we keep a null at the dog. Hence we'd need to solve the optimization problem

$$\textit{Maximize } [\textit{Average power at Squirrel}]\,,\ \textit{subject to Average Power at Dog = 0} \qquad (5)$$

In our case we would have a 4-dimensional optimization problem, corresponding to the 4 delay variables that we have control over. This would be be the most general and complete solution to the problem.

3. **Antenna Resolution:** The radio telescope dish at Arecibo, Puerto Rico operates at 10GHz and has a diameter of 300 meters (three football fields). If two distant radio sources (stars) are a light year ($\approx 10^{16}$m) apart from one another, and can each be seen distinctly through the Arecibo telescope, please provide an upper bound on their distance from Earth.

   State all approximations.

   Solution: *The beam width at target is* $2R\lambda/A$ *where* R *is the range to target,* λ *is the wavelength and* A *is the aperture size. For* $\lambda = 3 \times 10^8/10^10Hz = 3 \times 10^{-2}m$, $A = 300m$ *and* $Q = 10^16m$ *we have*

$$R = 300 \times 10^16/(2 \times 3 \times 10^{-2}) = 5 \times 10^{19}m \approx 5000LY$$

   For reference, the Milky Way is about 100KLY across.