

332:322

Principles of Communications Systems
Linear Systems Assessment Examination

Spring 2005

There are three questions. You have the class period to answer them. Place all answers on the test sheet. You may use whatever materials you'd like – including text books – but you're better off just trying to figure out everything on your own using first principles.

1. The Fourier transform of $x(t)$ is $X(f)$. Prove that the Fourier transform of $x(t - t_0)$ is $e^{-j2\pi ft_0}X(f)$.

SOLUTION:

$$\begin{aligned}\mathcal{F}\{x(t - t_0)\} &= \int_{-\infty}^{\infty} x(t - t_0)e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} x(v)e^{-j2\pi f(v+t_0)} dv \\ &= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} x(v)e^{-j2\pi fv} dv \\ &= e^{-j2\pi ft_0}X(f)\end{aligned}$$

2. Derive the convolution integral from first principles (as outlined in class) given by $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$, where $h(t)$ is the impulse response of a LTI system, $x(t)$ the input and $y(t)$ the output.

SOLUTION: *Convolution Integral: In time domain a linear system is described in terms of its impulse response which is defined as the response of the system (with zero initial conditions) to a unit impulse or delta function $\delta(t)$ applied to the input of a system. If the system is time invariant, then the shape of the impulse response is same no matter when the impulse is applied to the system.*

Let $h(t)$ denote the impulse response of a LTI system. Let this system be subjected to an arbitrary excitation $x(t)$. To determine the output $y(t)$ we first approximate the input $x(t)$ by staircase function composed of narrow rectangular pulses, each of duration $\Delta\tau$. The approximation becomes better for smaller $\Delta\tau$. As $\Delta\tau$ approaches zero, each pulse in the limit approaches a delta function weighed by a factor equal to the height of the pulse times $\Delta\tau$.

Consider a pulse which occurs at $t = n\Delta\tau$. By definition, the response of the system to a unit impulse or a delta function $\delta(t)$ occurring at $t = 0$ is $h(t)$. Due to the time invariance property the response of the system to a delta function weighed by a factor $x(n\Delta\tau)\Delta\tau$ occurring at $t = n\Delta\tau$, must be

$$\Delta y(t) = \lim_{\Delta\tau \rightarrow 0} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$

Above is the response to one component of input occurring at $t = n\Delta\tau$. Since the system is linear we can apply superposition principle to find the response to the sum of the input components (which together constitute $x(t)$) occurring at different times as,

$$y(t) = \lim_{\Delta\tau \rightarrow 0} \sum_{n=-\infty}^{n=\infty} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$

The right hand side of the above equation by definition is the convolution integral that is,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

3. The energy in a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Prove that if $X(f)$ is the Fourier transform of $x(t)$, then we must also have

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

You may use the fact that

$$\int_{-\infty}^{\infty} e^{j2\pi vt} dt = \delta(v)$$

SOLUTION: $|x(t)|^2 = x(t)x^*(t)$. We have

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df$$

so that

$$x^*(t) = \int_{-\infty}^{\infty} X^*(f)e^{-j2\pi ft} df$$

Thus,

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} X^*(f')e^{-j2\pi f't} df df' dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f)X^*(f') \left[\int_{-\infty}^{\infty} e^{j2\pi(f-f')t} dt \right] df df' \end{aligned}$$

where we've replaced f by f' to distinguish the dummy variable of integration in the integrals for $x(t)$ and $x^*(t)$. Then, using the fact yields

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f)X^*(f')\delta(f - f') df df' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f)X^*(f) df$$

via the sifting property. Recognizing that $X(f)X^*(f) = |X(f)|^2$ completes the proof.

4. Show that the overall system function $H(s)$ for the feedback system in FIGURE 1 is given by $H(s) = \frac{F(s)}{1 - F(s)G(s)}$.

SOLUTION:

$$W(s) = X(s) + G(s).Y(s), Y(s) = F(s).W(s) = F(s)[X(s) + G(s).Y(s)]$$

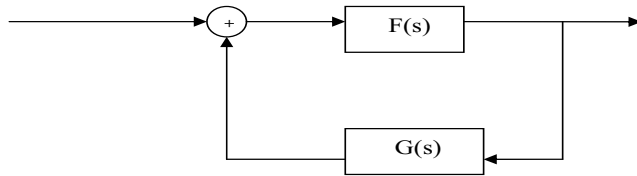
Re-arranging the previous equation,

$$F(s).X(s) = Y(s)[1 - F(s).G(s)]$$

System function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(s)}{1 - F(s).G(s)}$$

Figure 1: Feedback system for problem 4.



5. A signal $x(t)$ of finite energy is applied to a square-law device whose output is defined by $y(t) = x^2(t)$. The spectrum of $x(t)$ is limited to the frequency interval $-W \leq f \leq W$. Hence show that the spectrum of $y(t)$ is limited to $-2W \leq f \leq 2W$.

SOLUTION:

$$y(t) = x^2(t) = x(t)x(t)$$

Since multiplication in time domain corresponds to convolution in frequency domain, we may express the Fourier transform of $y(t)$ as

$$Y(f) = \int_{-\infty}^{\infty} X(\lambda)X(f - \lambda) d\lambda$$

where $X(f)$ is the Fourier transform of $x(t)$. However $X(f)$ is zero for $|f| > W$. Hence,

$$Y(f) = \int_{-W}^W X(\lambda)X(f - \lambda) d\lambda$$

In this integral we note that $X(f - \lambda)$ is limited to $-W \leq f - \lambda \leq W$. When $\lambda = -W$, we find that $-2W \leq f \leq 0$. When $\lambda = W$, $0 \leq f \leq 2W$. Accordingly the Fourier transform $Y(f)$ is limited to the frequency interval $-2W \leq f \leq 2W$.