Resource Allocation for Multicast in an OFDMA Network with Random Network Coding

Dan Zhang and Narayan Mandayam
WINLAB, Rutgers University
671 Route 1 South, North Brunswick, NJ 08902
{bacholic, narayan}@winlab.rutgers.edu

***Abstract*—In this paper, we consider multicast with Random Network Coding (RNC) over a wireless network using Orthogonal Frequency Division Multiple Access (OFDMA). Specifically, we propose a cross-layer resource allocation mechanism to minimize the total transmit power in the network to achieve a target throughput. The problem in its original form is a NP-hard mixed integer program. We alleviate this problem with a greedy power and subcarrier allocation algorithm that is combined with a node selection strategy that is enabled by RNC, which we refer to as "min-cut chasing." We compare it with a reference algorithm that assigns subcarriers independently based on the max-min fairness criterion followed by optimal power allocation. Our results reveal that the proposed greedy algorithm with min-cut chasing, which is of polynomial complexity, yields power savings of 3dB and is within 1dB of a lower bound based on an interference-free assumption.**

I. Introduction

We consider resource allocation in a wireless network that uses Orthogonal Frequency Division Multiple Access (OFDMA) and Random Network Coding (RNC) [1], [2] to support a multihop multicast at the target rate. This problem involves cross-layer optimization of subcarrier assignment, power allocation and flow design. Such a joint design problem is complicated by a number of factors: (i) The discrete nature of subcarrier assignment leads to a mixed integer program that is usually NP-hard; (ii) The discrete nature of bit assignment for OFDM symbols also leads to a NP-hard mixed integer program; (iii) Network flow design for optimal multicast with traditional routing is equivalent to a NP-hard Steiner tree packing problem [3]. Previous studies on OFDMA resource allocation use several approaches to alleviate these difficulties: (i) Subcarrier assignment that is designed separately, fixed a priori [4]–[6] or relaxed from discrete assignment to continuous assignment [7]; (ii) Relaxation of discrete bit assignment to continuous bit assignment [7], [8] or use of a greedy algorithm that simultaneously accomplishes subcarrier assignment and bit loading [4], [5], [8]; (iii) Flow design assuming the existence of a fixed multicast tree [6] or for simplified networking scenarios [9] [10].

In this paper, we further alleviate the flow design with RNC, which has been shown to achieve the cut-set bound and is especially amenable to practical implementation of multicast where the coding graphs constantly vary [11]. Because the cut-set bound can be described by linear constraints, it enables us to extend the greedy resource allocation algorithm to allow for multicast traffic with a target throughput over a multihop OFDMA network, with discrete subcarrier assignment and bit loading. In particular, we propose a technique that we refer to as "min-cut chasing," which successively provisions additional bits to the nodes that form the min-cut for the RNC multicast, and load them on the subcarriers that induce the minimum marginal power increase, in order to minimize the total transmit power. For comparison, we also propose a two-stage algorithm that first makes a near optimal max-min fair subcarrier assignment, then continuously and optimally loads the bits, which generalizes algorithms with equal or pre-determined subcarrier assignment [6] [8]. A relaxed version of the two-stage algorithm based on an interference-free assumption serves as a lower bound. Our algorithm shows significant power savings compared to the two-stage algorithm and performs close to the lower bound.

II. System Model

Consider a wireless network G consisting of a node set $\mathcal{M} = \{1, 2, \dots, M\}$, for which OFDMA is employed to support interference free transmissions. Our target is to build a multicast flow from node 1 to a set of destinations $\mathcal{D}$ at a required multicast throughput $\overline{r}$. We assume optimal RNC is employed for the course of the multicast. In what follows, we characterize the reception and interference models for this network as well as the throughput achieved via RNC.

A. Reception and Interference Model

The OFDMA scheme uses K subcarriers (from subcarrier 1 to subcarrier K) shared by all the nodes in $\mathcal{M}$ where each subcarrier width is assumed to be Δf, which is much less than the coherence bandwidth so that the channel response for subcarrier k from node i to j is modeled as a multiplicative scalar $H^k_{i,j}$. We assume nodes are stationary and hence the channels across the subcarriers can be frequency selective but time invariant. A fixed power budget $\overline{P}_i$ ($i \in \mathcal{M}$) is *optionally* imposed on each node.

In our reception model, we assume transmission from node i can be received successfully only by the nodes in a set $\mathcal{R}_i$. The transmit power of node i on subcarrier k is denoted as P^k_i. The maximum correctly decodable rate is given as [12]

$$r^k_{i,j} = \Delta f \log_2(1 + \beta P^k_i \rho^k_{i,j})(\text{bps}), \tag{1}$$

where

$$\rho_{i,j}^k = |H_{i,j}^k|^2/N_j \tag{2}$$

is the the normalized link gain with N_j being the receiver noise at node j. The constant $\beta \in (0,1)$ represents the SNR gap, which is the difference in required SNR between practical coding and the Shannon bound to achieve a target bit error rate (BER) [8] [12]. Define

$$\rho_i^k = \min_{j \in \mathcal{R}_i} \rho_{i,j}^k, \tag{3}$$

then node i can broadcast to all the nodes in $\mathcal{R}_i$ at the rate

$$r_i^k = \Delta f \log_2(1 + \beta P_i^k \rho_i^k)(\text{bps}). \tag{4}$$

Note the transmission cannot be received by any node not in $\mathcal{R}_i$.

In our interference model, for every node i, there is a set $\mathcal{F}_i$ consisting of all the nodes that would interfere with node i. Therefore under no circumstance can node i use the same subcarrier as any node in $\mathcal{F}_i$. The interference model captures a number of practical issues. For example, in order to avoid the hidden node problem, some nodes cannot transmit at the same time. The interference set $\mathcal{F}_i$ need not be disjoint from the reception set $\mathcal{R}_i$. A subcarrier assignment consists of a collection of sets $\{S_i\}_{i=1}^M$, where $S_i \subset \{1, 2, \ldots, K\}$ represents the subcarriers allocated to node i. A feasible allocation scheme should satisfy the power constraints

$$\sum_{k \in S_i} P_i^k \le \overline{P}_i, \quad \forall 1 \le i \le M \tag{5}$$

and the interference constraints

$$S_i \cap S_j = \emptyset, \quad \forall i \in \mathcal{M}, j \in \mathcal{F}_i. \tag{6}$$

With a feasible allocation, the transmission rate of node i is given by

$$r_i = \sum_{k \in S_i} r_i^k. \tag{7}$$

B. *Hypergraph Model for Random Network Coding*

In the hypergraph model for analyzing RNC (first introduced in [13] and later expanded in [14]), the wireless network G can be modeled as a hypergraph $G = (\mathcal{M}, \mathcal{E})$, where the set $\mathcal{E}$ of hyperarcs is defined as $\mathcal{E} = \{(i, \mathcal{R}_i) | i \in \mathcal{M}\}$, which follows from our reception model. Once the underlying MAC (subcarrier assignment) and LINK (power allocation) layers are given, each node i would be transmitting at λ_i packets per second, which is considered as the capacity of the hyperarc $(i, \mathcal{R}_i)$.

Given two nodes s and t, a cut for the pair (s,t) is defined to be a set $\mathcal{T}$ of nodes such that $t \in \mathcal{T}$ and $s \in \mathcal{T}^c$. The collection of all cuts for (s,t) is denoted by $C(s,t)$. The size of $\mathcal{T}$ is defined as

$$c(\mathcal{T}) = \sum_{\substack{i \in \mathcal{T}^c \\ \mathcal{R}_i \cap \mathcal{T} \neq \emptyset}} \lambda_i. \tag{8}$$

The min cut $\mathcal{T}_{\min}$ for (s,t), whose size is denoted as $c_{\min}(s,t)$ is a cut satisfying

$$c(\mathcal{T}_{\min}) = c_{\min}(s,t) = \min_{\mathcal{T}' \in C(s,t)} c(\mathcal{T}'). \tag{9}$$

It has been shown [13] [14] that for a RNC based multicast, the maximum multicast throughput (the highest rate at which nodes in $\mathcal{D}$ can receive information from the source simultaneously) is given by the cut-set bound

$$\min_{d \in \mathcal{D}} c_{\min}(1,d) = \min_{d \in \mathcal{D}} \min_{\mathcal{T}' \in C(1,d)} \sum_{\substack{i \in \mathcal{T}'^c \\ \mathcal{R}_i \cap \mathcal{T}' \neq \emptyset}} \lambda_i. \tag{10}$$

It would be useful in what follows to identify G with a digraph $G' = (\mathcal{M}, \mathcal{E}')$, where $\mathcal{E}' = \{(i,j) | i \in \mathcal{M}, j \in \mathcal{R}_i\}$, and let $\{c_{i,j}\}$ be the capacities of the links in $\mathcal{E}'$. Then the size of the cut $\mathcal{T} \in C(s,t)$, denoted as $c^{\text{dir}}(\mathcal{T})$, can be defined as

$$c^{\text{dir}}(\mathcal{T}) = \sum_{i \in \mathcal{T}^c} \sum_{j \in \mathcal{R}_i \cap \mathcal{T}} c_{i,j}. \tag{11}$$

The min cut for (s,t) in G', denoted as $c_{\min}^{\text{dir}}(s,t)$, is defined as

$$c_{\min}^{\text{dir}}(s,t) = \min_{\mathcal{T} \in C(s,t)} c^{\text{dir}}(\mathcal{T}). \tag{12}$$

Assume each packet has a fixed length of L bits, then

$$\lambda_i L = r_i. \tag{13}$$

Therefore, we may state our target multicast throughput constraint as

$$c_{\min}(1,d) = \min_{\mathcal{T}' \in C(s,t)} \sum_{\substack{i \in \mathcal{T}'^c \\ \mathcal{T}' \cap \mathcal{F}_i \neq \emptyset}} r_i \ge \overline{r}, \quad \forall d \in \mathcal{D}, \tag{14}$$

or equivalently

$$\sum_{\substack{i \in \mathcal{T}'^c \\ \mathcal{T}' \cap \mathcal{F}_i \neq \emptyset}} r_i \ge \overline{r}, \quad \forall d \in \mathcal{D}, \forall \mathcal{T}' \in C(1,d). \tag{15}$$

III. Resource Allocation for Multicast in a Multihop OFDMA Network Using RNC

Minimum cost multicast using RNC was first outlined in [13], then elaborated in [11] and later discussed in [15] and [16]. These previous works provide a framework as well as optimization strategies that are based on utility functions and cost metrics in a generic wireless network. In this paper, we specifically consider the resource allocation (subcarrier assignment and power allocation) in a OFDMA network with multihop multicast using RNC, to achieve a target throughput

$\overline{r}$. The basic problem can be stated as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{M} P_i, && \text{(16a)}\\
\text{subject to} \quad & P_i = \sum_{k\in S_i} P_i^k, \quad P_i^k \geq 0 && \text{(16b)}\\
& S_i \cap S_j = \emptyset, \quad \forall i \text{ and } j \in \mathcal{R}_i, && \text{(16c)}\\
& r_i \leq \Delta f \sum_{k\in S_i} \log_2(1+\beta P_i^k \rho_i^k), && \text{(16d)}\\
& c_{\min}(1,d) \geq \overline{r}, \quad \forall d \in \mathcal{D}. && \text{(16e)}\\
\text{variables} \quad & \{P_i^k\}, \{S_i\}, \{r_i\}.
\end{aligned}
$$

It is clear that, though we have a linear objective function (16a), linear power allocation (16b) and linear RNC throughput constraint (16e), the bit assignment constraint (16d) is not convex because every OFDM symbol can only contain an integer number of bits, i.e., $r_i^k/\Delta f \in \mathbb{Z}$, which makes (16) a nonlinear knapsack problem [17]. It becomes convex if real (continuous) bits are assumed. But even with this assumption, the subcarrier assignment constraint (16c) is not convex. In fact, it is discrete in terms of subcarriers and complicated by the interference model. Such an interference model has also been considered for the downlink of a cellular system [7], [8] and for a one-hop network [5], but neither considers RNC or multicast.

The nonconvexity difficulty incurred by subcarrier assignment can be tackled with two strategies, both leading to suboptimal solutions. The first strategy begins by noticing that, with the real bit assignment assumption, once a feasible subcarrier assignment is obtained, (16c) can be removed from (16), which then becomes a convex program. In other words, this strategy can be executed through two stages. In the first stage, a feasible subcarrier assignment is determined independently based on some optimality criterion. For example, we first assign subcarriers based on the max-min fairness criterion such that the smallest number of subcarriers assigned to any node is maximized. Then a convex program is solved to determine the optimal power allocation. Although this strategy seems reasonable giving as many subcarriers to every node as possible, it disregards the channel state of the subcarriers and the topological connectivity of the nodes. The second strategy, which has been more widely used, is the greedy resource allocation algorithm [4], [5], [8] based on integer bit loading. It jointly allocates integer bits, discrete subcarriers and power to different nodes iteratively until all the constraints are satisfied. In this paper we will adopt the greedy strategy but also discuss the two-stage strategy for comparison.

A. *Greedy Algorithm to Solve* (16)

The greedy resource allocation strategy successively chooses a particular node and a particular subcarrier, then assigns additional bits to them. The choice should minimize the induced power increase, and at the same time satisfy (16c). Specifically, loading additional bits on the subcarrier implies increasing the modulation efficiency of an OFDM symbol (e.g., larger constellation for MQAM). Assume one bit is loaded (i.e., $\Delta r_i^k/\Delta f = 1$) on to subcarrier k of node i in an iteration, then the power increase ΔP_i^k can be calculated using (4) as

$$
\Delta P_i^k = \frac{2^{\frac{r_i^k}{\Delta f}+1}-1}{\beta P_i^k \rho_i^k} - \frac{2^{\frac{r_i^k}{\Delta f}}-1}{\beta P_i^k \rho_i^k}. \tag{17}
$$

Since we ultimately want to build a multicast at the target throughput $\overline{r}$, as shown in (16e), the nodes and subcarriers that the algorithm chooses should contribute to this goal. Therefore it makes sense to choose from the nodes whose transmit rates affect the smallest min-cut. Specifically, let

$$
\mathcal{D}_{\min} = \arg\min_{d\in\mathcal{D}} c_{\min}(1,d), \tag{18}
$$

define

$$
\begin{aligned}
\mathcal{C}(d) = \{ i \in \mathcal{M} | \exists \mathcal{T} \in C(1,d) \text{ s.t. } \mathcal{R}_i \cap \mathcal{T} \neq \emptyset \\
\text{and } c(\mathcal{T}) = c_{\min}(1,d) \}, \quad \forall d \in \mathcal{D}_{\min},
\end{aligned} \tag{19}
$$

and

$$
\mathcal{C}(\mathcal{D}_{\min}) = \bigcup_{d\in\mathcal{D}_{\min}} \mathcal{C}(d). \tag{20}
$$

Then we would assign an additional bit to one of the nodes in $\mathcal{C}(\mathcal{D}_{\min})$. We refer to the strategy that always picks a node from $\mathcal{C}$ using (18)–(20) as "min-cut chasing" (MCC). The basic greedy resource allocation algorithm with MCC is shown in Alg. 1 with a description of steps in italics.

Alg. 1 Greedy Resource Allocation with Ideal MCC

Require: $S_i = \emptyset, \mathcal{C} = \{1\}, P_i^k = 0, r_i = 0$, calculate ΔP_i^k according to (17)

repeat
- $\mathcal{D}_{\min} \leftarrow \arg\min_{d\in\mathcal{D}} c_{\min}(1,d)$ {*pick the smallest flow*}
- calculate $\mathcal{C}(\mathcal{D}_{\min}) = \cup_{d\in\mathcal{D}_{\min}} \mathcal{C}(d)$ using (19) {*MCC*}
- $(i_{\min}, k_{\min}) \leftarrow \arg\min_{i\in\mathcal{C}(\mathcal{D}_{\min}), k\in S_i} \Delta P_i^k$ {*greedy allocation*}
- $r_{i_{\min}} \leftarrow r_{i_{\min}} + \Delta f$ {*1-bit loading*}
- $P_{i_{\min}}^{k_{\min}} \leftarrow P_{i_{\min}}^{k_{\min}} + \Delta P_{i_{\min}}^{k_{\min}}$
- $S_j \leftarrow S_j \setminus \{k_{\min}\}, \quad \forall j \in \mathcal{F}_{i_{\min}}$ {*avoid interference*}
- update $\Delta P_{i_{\min}}^{k_{\min}}$ according to (17)
- calculate r_i according to (7)

until $\min_d c_{\min}(1,d) \geq \overline{r}$,

B. *Min-Cut Chasing*

Execution of MCC via evaluation of (20) is not practical. Although the constraints it represents are all linear, as shown in (15), the number of the constraints grows exponentially with M. Instead we chase a reasonable superset of $\mathcal{C}$ using an idea that was suggested in [13]. Specifically, for the hypergraph model G of our network, the min-cut $c_{\min}(1,d)$ for a given destination d can be found as the max flow of the corresponding digraph G' discussed in Section II-B. If $\{f_{i,j}^d\}$ designate a network flow from node 1 to d in G', where link

(i,j) carries the flow $f_{i,j}^d$, then the max-flow can be solved using a linear program (LP) as follows

Theorem 1: The min-cut $c_{\min}$ as shown in (14) can be calculated as the optimal value of the following LP

$$\begin{aligned}
\text{maximize} \quad & \sum_{j\in\mathcal{R}_1} f_{1,j}^d - \sum_{1\in\mathcal{R}_j} f_{j,1}^d, && \text{(21a)}\\
\text{subject to} \quad & \sum_{j\in\mathcal{R}_i} f_{i,j}^d - \sum_{i\in\mathcal{R}_j} f_{j,i}^d = 0, \quad \forall i \neq 1, d, && \text{(21b)}\\
& \sum_{j\in\mathcal{R}_i} f_{i,j}^d \leq r_i, \quad \forall i \in \mathcal{M}\setminus\{d\}, && \text{(21c)}\\
& f_{i,j}^d \geq 0, \quad \forall i \in \mathcal{M} \text{ and } j \in \mathcal{R}_i, && \text{(21d)}\\
\text{variables} \quad & \{f_{i,j}\}.
\end{aligned}$$

The number of constraints in LP (21) is $3M-3$ and hence can be solved efficiently. We first prove a weak duality result:

Lemma 1: Let f^{d*} denote the optimal value of (21). We have

$$f^{d*} \leq c_{\min}(1,d). \tag{22}$$

Proof to Lemma 1: For any i such that $\mathcal{R}_i \neq \emptyset$, pick an arbitrary $j_i \in \mathcal{R}_i$. Add capacity $c_{i,j}$ to link (i,j) where

$$c_{i,j} = \begin{cases} f_{i,j}^{d*}, & j \neq j_i,\\ r_i - \sum_{j\in\mathcal{R}_i\setminus\{j_i\}} f_{i,j}^{d*}, & j = j_i. \end{cases} \tag{23}$$

Then consider the max-flow problem on the digraph G' where $\{f_{i,j}^d\}$ designate a flow from node 1 to d:

$$\begin{aligned}
\text{maximize} \quad & \sum_{j\in\mathcal{R}_1} f_{1,j}^d - \sum_{1\in\mathcal{R}_j} f_{j,1}^d, && \text{(24a)}\\
\text{subject to} \quad & \sum_{j\in\mathcal{R}_i} f_{i,j}^d - \sum_{i\in\mathcal{R}_j} f_{j,i}^d = 0, \quad \forall i \neq 1, d, && \text{(24b)}\\
& f_{i,j}^d \leq c_{i,j}, \quad \forall i \in \mathcal{M}\setminus\{d\} \text{ and } j \in \mathcal{R}_i, && \text{(24c)}\\
& f_{i,j}^d \geq 0, \quad \forall i \in \mathcal{M} \text{ and } j \in \mathcal{R}_i, && \text{(24d)}\\
\text{variables} \quad & \{f_{i,j}^d\}.
\end{aligned}$$

Since the feasible region of (24) is a subset of that of (21), the optimal value of (24), denoted as f^{d**}, satisfies $f^{d**} \leq f^{d*}$. Since $f_{i,j}^{d*}$ satisfy all the constraints of (24), it follows that $f^{d**} = f^{d*}$. By the weak duality of a capacitated digraph, $\forall \mathcal{T} \in C(1,d)$ we have

$$\begin{aligned}
f^{d*} &= f^{d**} \leq c^{\text{dir}}(\mathcal{T})\\
&= \sum_{i\in\mathcal{T}^c} \sum_{j\in\mathcal{R}_i\cap\mathcal{T}} c_{i,j} \leq \sum_{\substack{i\in\mathcal{T}^c\\ \mathcal{R}_i\cap\mathcal{T}\neq\emptyset}} r_i \leq c(\mathcal{T}).
\end{aligned} \tag{25}$$

Since (25) holds for any cut $\mathcal{T} \in C(1,d)$, it also holds for the min-cut, hence $f^{d*} \leq c_{\min}(1,d)$. ■

In order to prove Theorem 1, we also need:

Claim 1: There exists an optimizer $\{f_{i,j}^{d*}\}$ to (21) that has the following properties: (i) There is no flow in any cycle; (ii) There is no outgoing flow from node d; (iii) The flow designated by $\{f_{i,j}^{d*}\}$ can be decomposed into a finite number of flows, each carried by a simple path from node 1 to node d; (iv) It constitutes a max-flow on G' (cf. (24)) whose capacities are given as in (23).

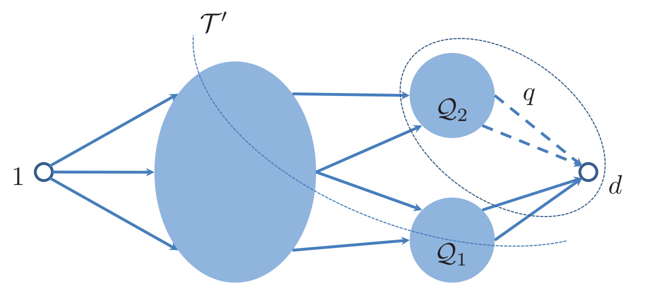

Fig. 1. Illustration of the proof to Theorem 1. Note $\mathcal{Q}_1$ and $\mathcal{Q}_2$ all have forward links to d. $\mathcal{Q}_1$ has zero residual capacity and $\mathcal{Q}_2$ has positive residual capacity. A (min-)cut $\mathcal{T}'$ for $(1,d)$ contracts to a (min-)cut for $(1,q)$.

Proof: Consider the following LP:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{R}_i} f_{i,j}^d, && \text{(26a)}\\
\text{subject to} \quad & \sum_{j\in\mathcal{R}_i} f_{i,j}^d - \sum_{i\in\mathcal{R}_j} f_{j,i}^d = 0, \quad \forall i \neq 1, d, && \text{(26b)}\\
& \sum_{j\in\mathcal{R}_1} f_{1,j}^d - \sum_{1\in\mathcal{R}_j} f_{j,1}^d = f^{d*}, && \text{(26c)}\\
& \sum_{j\in\mathcal{R}_i} f_{i,j}^d \leq r_i, \quad \forall i \in \mathcal{M}, && \text{(26d)}\\
& f_{i,j}^d \geq 0, \quad \forall i \in \mathcal{M} \text{ and } j \in \mathcal{R}_i, && \text{(26e)}\\
\text{variables} \quad & \{f_{i,j}^d\}.
\end{aligned}$$

Comparison of (21) and (26) reveals that the optimizer of (26) is also an optimizer of (21) with the total link flows minimized. Property (i) follows from the fact that removal of any flow in a cycle can only lower the objective of (26) without violating any constraints; Property (ii) follows from property (i) from the fact that an outgoing flow from d eventually returns to d through a cycle due to flow conservation; Property (iii) follows from a standard flow decomposition argument [18]; Property (iv) follows from the proof to Lemma 1. ■

Proof to Theorem 1: In light of Lemma 1, all we need to do is to find a $\mathcal{T} \in C(1,d)$, such that $f^{d*} = c(\mathcal{T})$. Since $c(\mathcal{T}) \geq c_{\min}(1,d)$, we would have then established $f^{d*} = c_{\min}(1,d)$. We show such a $\mathcal{T} \in C(1,d)$ exists by induction. When $\mathcal{M} = \{1,d\}$, $\mathcal{T} = \{d\}$ satisfies $f^{d*} = c(\mathcal{T})$. Assume for $|\mathcal{M}| < M$, a $\mathcal{T} \in C(1,d)$ always exists such that $c(\mathcal{T}) = f^{d*}$. Then we need to show that such a $\mathcal{T} \in C(1,d)$ also exists when $|\mathcal{M}| = M$.

With the flow designated by $\{f_{i,j}^{d*}\}$ on G', consider the set of all nodes $\mathcal{Q}$ that have d as a recipient, i.e., $\mathcal{Q} = \{i | d \in \mathcal{R}_i\}$. If $1 \in \mathcal{Q}$, let $\mathcal{T} = \{d\}$ and the proof is complete. Otherwise we may assume that $\forall i \in \mathcal{Q}$, all outgoing flow is only through the link (i,d), i.e.,

$$f_{i,j}^{d*} = 0, \forall j \in \mathcal{R}_i \text{ and } j \neq d. \tag{27}$$

Suppose it were not so, then we show how the flows $\{f_{i,j}^{d*}\}$ and link capacities $\{c_{i,j}\}$ can be modified such that $\{f_{i,j}^{d*}\}$ still constitute a max-flow of G' with the new capacities, while still satisfying the properties in Claim 1. If $f_{i,j}^{d*} > 0$ for some $j \neq d$ and $j \in \mathcal{R}_i$, then a simple path $p = (s_1 = i, s_2 = j, s_3, \ldots, s_h = d)$ exists, which carries a positive flow δf (it follows from property (i) and (iii) of Claim 1). We can reassign the flow δf from p onto the link (i, d), i.e., $f_{s_t, s_{t+1}}^{d*} \leftarrow f_{s_t, s_{t+1}}^{d*} - \delta f$, $t = 1, 2, \ldots, h-1$ and $f_{i,d}^{d*} \leftarrow f_{i,d}^{d*} + \delta f$, and reset the capacities $\{c_{i,j}\}$ $(i \in \mathcal{Q})$ of the digraph G'

$$c_{i,j} = \begin{cases} 0, & j \neq d, \\ r_i, & j = d. \end{cases} \tag{28}$$

It can be verified that the resulting $\{f_{i,j}^{d*}\}$ are still an optimizer of (21) with the four properties in Claim 1. The procedure can be repeated for all nodes in $\mathcal{Q}$ until (27) holds.

Consider the partition $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$, such that $\mathcal{Q}_1$ consists of all the nodes whose outgoing links have no residual capacity (cf. (28)), i.e.,

$$\begin{aligned} \mathcal{Q}_1 &= \{i \in \mathcal{Q} | f_{i,d}^{d*} = c_{i,d} = r_i\}, \\ \mathcal{Q}_2 &= \{i \in \mathcal{Q} | f_{i,d}^{d*} < c_{i,d} = r_i\}. \end{aligned} \tag{29}$$

The partition is shown in Fig. 1, where we use dashed arrows to represent links with positive residual capacities and solid arrows for zero residual capacities. Let us contract $\{d\} \cup \mathcal{Q}_2$ into a compound node q. Due to property (ii) of Claim 1 and (27), q has no outgoing flows. Consequently, the contraction produces a new hypergraph $\widetilde{G} = (\widetilde{\mathcal{M}}, \widetilde{\mathcal{E}})$ with

$$\begin{aligned} \widetilde{\mathcal{M}} &= (\mathcal{M} \setminus \{d\} \setminus \mathcal{Q}_2) \cup \{q\}, \\ \widetilde{\mathcal{E}} &= \{(i, \widetilde{\mathcal{R}}_i) | i \in \widetilde{\mathcal{M}}\}, \end{aligned} \tag{30}$$

where $\widetilde{\mathcal{R}}_q = \emptyset$ and $\forall i \in \widetilde{\mathcal{M}} \setminus \{q\}$,

$$\widetilde{\mathcal{R}}_i = \begin{cases} \mathcal{R}_i, & \mathcal{R}_i \cap (\{d\} \cup \mathcal{Q}_2) = \emptyset, \\ (\mathcal{R}_i \setminus \{d\} \setminus \mathcal{Q}_2) \cup \{q\}, & \mathcal{R}_i \cap (\{d\} \cup \mathcal{Q}_2) \neq \emptyset. \end{cases} \tag{31}$$

The hyperarc capacity $\{\widetilde{r}_i\}$ for $\widetilde{G}$ remain the same for $i \in \widetilde{\mathcal{M}} \setminus \{q\}$, i.e.,

$$\widetilde{r}_i = r_i, \quad \forall i \in \widetilde{\mathcal{M}} \setminus \{q\}. \tag{32}$$

The contraction also produces a new digraph $\widetilde{G}' = (\widetilde{\mathcal{M}}, \widetilde{\mathcal{E}}')$ with

$$\widetilde{\mathcal{E}}' = \{(i, j) | i \in \widetilde{\mathcal{M}}, j \in \widetilde{\mathcal{R}}_i\}. \tag{33}$$

A new flow on $\widetilde{G}'$ is given by combining the flows into $\{d\} \cup \mathcal{Q}_2$. Specifically, $\forall i \in \widetilde{\mathcal{M}} \setminus \{q\}$, we let

$$\widetilde{f}_{i,j}^d = \begin{cases} f_{i,j}^{d*}, & j \neq q, \\ \sum_{j' \in \mathcal{R}_i \cap (\{d\} \cup \mathcal{Q}_2)} f_{i,j'}^{d*}, & j = q, \end{cases}, \forall i \in \widetilde{\mathcal{M}} \setminus \{q\}. \tag{34}$$

It can be verified that the flow designated by $\{\widetilde{f}_{i,j}^d\}$ satisfies (21b)((24b)) and its value remains to be f^{d*}, i.e.,

$$\sum_{j \in \widetilde{\mathcal{R}}_1} \widetilde{f}_{1,j}^d - \sum_{1 \in \widetilde{\mathcal{R}}_j} \widetilde{f}_{j,1}^d = f^{d*}. \tag{35}$$

A new set of link capacities is given by combining the capacities of links into $\{d\} \cup \mathcal{Q}_2$,

$$\widetilde{c}_{i,j} = \begin{cases} c_{i,j}, & j \neq q, \\ \sum_{j' \in \mathcal{R}_i \cap (\{d\} \cup \mathcal{Q}_2)} c_{i,j'}, & j = q, \end{cases} \forall i \in \widetilde{\mathcal{M}} \setminus \{q\}. \tag{36}$$

It can be verified that $\{\widetilde{f}_{i,j}^d\}$, $\{\widetilde{r}_i\}$ satisfy (21c), and that $\{\widetilde{f}_{i,j}^d\}$, $\{\widetilde{c}_{i,j}\}$ satisfy (24c).

Next we construct the desired cut $\mathcal{T} \in C(1, d)$ in G by identifying it with a cut $\widetilde{\mathcal{T}} \in C(1, q)$ in $\widetilde{G}$, through the following steps:

1) With (36), it can be verified (see Fig. 1) that $\forall \mathcal{T}' \in C(1, d)$ in G', such that $\mathcal{Q}_2 \subset \mathcal{T}'$, we have $c^{\text{dir}}(\mathcal{T}') = c^{\text{dir}}(\widetilde{\mathcal{T}}')$ where $\widetilde{\mathcal{T}}' \in C(1, q)$ in $\widetilde{G}'$ denotes $\mathcal{T}'$ after contraction. With (32), it can also be verified that $c(\mathcal{T}') = c(\widetilde{\mathcal{T}}')$.
2) For the digraph min-cut $\mathcal{T}' \in C(1, d)$ in G', because all the links from $\mathcal{Q}_2$ to d have positive residual capacity (cf. (29)), $\mathcal{Q}_2 \subset \mathcal{T}'$. Therefore it can be contracted to $\widetilde{\mathcal{T}}' \in C(1, q)$ in $\widetilde{G}'$.
3) By the max-flow min-cut theorem $c^{\text{dir}}(\mathcal{T}') = f^{d*}$. It follows from step 1 that $c^{\text{dir}}(\widetilde{\mathcal{T}}') = f^{d*}$.
4) Based on step 3 and (35), the max-flow min-cut theorem implies that $\{\widetilde{f}_{i,j}^d\}$ designate a max-flow in $\widetilde{G}'$. Thus $\{\widetilde{f}_{i,j}^d\}$ is an optimizer of (24) for $\widetilde{G}'$ with the optimal value f^{d*}. By the proof of Lemma 1, it is also an optimizer of (21) with the optimal value f^{d*}.
5) By step 4 and the induction assumption, a hypergraph min-cut $\widetilde{\mathcal{T}} \in (1, q)$ in $\widetilde{G}$ exists, such that $f^{d*} = c(\widetilde{\mathcal{T}})$. By step 1, $f^{d*} = c(\mathcal{T})$ where $\mathcal{T} \in C(1, d)$ in G is $\widetilde{\mathcal{T}}'$ before contraction. This concludes the proof.

■

Theorem 1 also suggests an alternative MCC procedure that is more amenable to implementation. We first obtain $\mathcal{D}_{\min}$ by (18) where $c_{\min}(1, d)$ is calculated through (21). Once $c_{\min}(1, d)$ $(d \in \mathcal{D}_{\min})$ is known, we solve (26) with $f^{d*} = c_{\min}(1, d)$. Then instead of using (19) and (20), we chase the nodes in the set

$$\mathcal{C}'(\mathcal{D}_{\min}) = \cup_{d \in \mathcal{D}_{\min}} \mathcal{C}'(d). \tag{37}$$

where

$$\mathcal{C}'(d) = \{i \in \mathcal{M} \setminus \{d\} | \sum_{j \in \mathcal{R}_i} f_{i,j}^{d*} = r_i\}. \tag{38}$$

The application of (26) is merely to remove flows in cycles that could unnecessarily enlarge $\mathcal{C}'(d)$. Let $i \in \mathcal{C}(d)$ $(d \in \mathcal{D}_{\min})$ and $\mathcal{T} \in C(1, d)$ such that $c(\mathcal{T}) = c_{\min}(1, d)$, we have

$$\begin{aligned} f^{d*} = c_{\min}(1, d) &= \sum_{\substack{i \in \mathcal{T}^c \\ \mathcal{R}_i \cap \mathcal{T} = \emptyset}} r_i \\ &\geq \sum_{\substack{i \in \mathcal{T}^c \\ \mathcal{R}_i \cap \mathcal{T} = \emptyset}} \sum_{j \in \mathcal{R}_i} f_{i,j}^* \geq \sum_{i \in \mathcal{T}^c} \sum_{j \in \mathcal{R}_i \cap \mathcal{T}} f_{i,j}^{d*} \geq f^{d*}, \end{aligned} \tag{39}$$

where the last inequality follows from flow conservation, i.e., that any incident flow across a cut is greater than the net incident flow across the cut. Eq. (39) implies that $\sum_{j\in\mathcal{R}_i} f_{i,j}^{d*} = r_i$, hence $\mathcal{C}(d) \subset \mathcal{C}'(d)$ and $\mathcal{C}(\mathcal{D}_{\min}) \subset \mathcal{C}'(\mathcal{D}_{\min})$. If we keep allocating power to the nodes in $\mathcal{C}'(\mathcal{D}_{\min})$, eventually we would be allocating power to nodes in $\mathcal{C}(\mathcal{D}_{\min})$. By (18), it follows that $\min_{d\in\mathcal{D}} c_{\min}(1,d)$ will be increased, as well as the achievable throughput for RNC (cf. (10) and (18)).

Another implication of Theorem 1 is that the original problem (21) can be equivalently solved by the following problem:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{M} P_i, && \text{(40a)}\\
\text{subject to} \quad & P_i = \sum_{k\in S_i} P_i^k, \quad P_i^k \geq 0 && \text{(40b)}\\
& S_i \cap S_j = \emptyset, \quad \forall i \text{ and } j \in \mathcal{R}_i, && \text{(40c)}\\
& r_i \leq \Delta f \sum_{k\in S_i} \log_2(1+\beta P_i^k \rho_i^k), && \text{(40d)}\\
& \forall d \in \mathcal{D}: \\
& \sum_{j\in\mathcal{R}_i} f_{i,j}^d - \sum_{i\in\mathcal{R}_j} f_{j,i}^d = 0, \quad \forall i \neq 1, d && \text{(40e)}\\
& \sum_{j\in\mathcal{R}_i} f_{i,j}^d \leq r_i, \quad \forall i, && \text{(40f)}\\
& f_{i,j}^d \geq 0, \quad \forall i \text{ and } j \in \mathcal{R}_i, && \text{(40g)}\\
& \sum_{j\in\mathcal{R}_1} f_{1,j}^d - \sum_{1\in\mathcal{R}_j} f_{j,1}^d \geq \overline{r}, && \text{(40h)}\\
\text{variables} \quad & \{P_i^k\}, \{S_i\}, \{r_i\}, \{f_{i,j}\}.
\end{aligned}$$

Eq. (40e)–(40g) are adapted from (21) to replace (16e) and to allow for MCC. Once a bit is greedily loaded, a set of new rates $\{r_i\}$ is plugged into (40e)–(40g) (or (21)) to start chasing for the next min-cut using (37). Alg. 2 shows the details.

Alg. 2 Greedy Resource Allocation with Alternative MCC

Require: $S_i = \{1,\ldots,K\}, \mathcal{C}' = \{1\}, P_i^k = 0, r_i = 0$, calculate ΔP_i^k according to (17)

repeat
  solve (21) for f^{d*}, $\forall d \in \mathcal{D}$ {*Theorem 1*}
  $\mathcal{D}_{\min} \leftarrow \arg\min_{d\in\mathcal{D}} c_{\min}(1,d)$ {*pick the smallest flow*}
  $\mathcal{C}'(\mathcal{D}_{\min}) = \cup_{d_{\min}\in\mathcal{D}_{\min}} \mathcal{C}'(d_{\min})$ using (37) {*MCC*}
  $(i_{\min}, k_{\min}) \leftarrow \arg\min_{i\in\mathcal{C}', k\in S_i} \Delta P_i^k$ {*greedy allocation*}
  $r_{i_{\min}}^{k_{\min}} \leftarrow r_{i_{\min}}^{k_{\min}} + \Delta f$ {*1-bit loading*}
  $P_{i_{\min}}^{k_{\min}} \leftarrow P_{i_{\min}}^{k_{\min}} + \Delta P_{i_{\min}}^{k_{\min}}$
  $S_j \leftarrow S_j \setminus \{k_{\min}\}, \quad \forall j \in \mathcal{F}_{i_{\min}}$ {*avoid interference*}
  update $\Delta P_{i_{\min}}^{k_{\min}}$ according to (17)
  calculate $r_{i_{\min}}$ according to (7)
until $\min_d c_{\min}(1,d) \geq \overline{r}$,

IV. Resource Allocation with Power Constraint

With power constraint, problem (16) can be equivalently formulated, with the help of Theorem 1, as

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{M} P_i, && \text{(41a)}\\
\text{subject to} \quad & P_i = \sum_{k\in S_i} P_i^k, \quad P_i^k \geq 0 && \text{(41b)}\\
& P_i \leq \overline{P}_i, && \text{(41c)}\\
& S_i \cap S_j = \emptyset, \quad \forall i \text{ and } j \in \mathcal{R}_i, && \text{(41d)}\\
& r_i \leq \Delta f \sum_{k\in S_i} \log_2(1+\beta P_i^k \rho_i^k), && \text{(41e)}\\
& \forall d \in \mathcal{D}: \\
& \sum_{j\in\mathcal{R}_i} f_{i,j}^d - \sum_{i\in\mathcal{R}_j} f_{j,i}^d = 0, \quad \forall i \neq 1, d && \text{(41f)}\\
& \sum_{j\in\mathcal{R}_i} f_{i,j}^d \leq r_i, \quad \forall i \in \mathcal{M}, && \text{(41g)}\\
& \sum_{j\in\mathcal{R}_1} f_{1,j}^d - \sum_{1\in\mathcal{R}_j} f_{j,1}^d \geq \overline{r}, && \text{(41h)}\\
& f_{i,j}^d \geq 0, \quad \forall i \text{ and } j \in \mathcal{R}_i, && \text{(41i)}\\
\text{variables} \quad & \{P_i^k\}, \{S_i\}, \{r_i\}, \{f_{i,j}\}.
\end{aligned}$$

Note (41c) is the additional individual constraint on power budget. If an optimal/suboptimal solution calculated by Alg. 2 does not violate these individual power constraints, it is also an optimal/suboptimal solution to (41). Therefore, when Alg. 2 fails to produce a feasible solution to (41), an alternative greedy algorithm is required. A common approach (e.g., [5]) to incorporate the individual power constraints is by replacing the greedy allocation step in Alg. 2 with

$$(i_{\min}, k_{\min}) \leftarrow \arg\min_{i\in\mathcal{C}'(d_{\min}), k\in S_i} \Delta P_i^k / \overline{P}_i. \tag{42}$$

The rationale for this change is that, with the individual power constraint, the minimum marginal power increase may not represent the best choice since nodes with small power budgets may still use up all the power quickly. Choosing the minimum marginal percentage increase allows for such possibility, and hence represents a better option if Alg. 2 does not yield a feasible solution. This procedure may fail to proceed if none of the nodes identified by MCC can be loaded with an additional bit without violating the power constraint. Details of the modified algorithm are shown in Alg. 3.

V. Reference Algorithms

In this section, we further propose two reference algorithms to be compared with the greedy resource allocation algorithms, with or without individual power constraints. The basis of these algorithms follows from the observation that, if we remove (40c) from (40) (or (41d) from (41)) and independently determine subcarrier assignment $\{S_i\}$, then we only need to solve the remaining convex program for optimal power

Alg. 3 Greedy Resource Allocation with MCC and Individual Power Constraints

Require: $S_i = \{1, \ldots, K\}(\forall i \in \mathcal{M}), \mathcal{C}' = \{1\}, P_i^k = 0, r_i = 0$, calculate ΔP_i^k according to (17)

repeat
  solve (21) for f^{d*}, $\forall d \in \mathcal{D}$ {Theorem 1}
  $d_{\min} \leftarrow \arg\min_{d \in \mathcal{D} \setminus \mathcal{B}} c_{\min}(1, d)$ {pick the smallest flow}
  calculate $\mathcal{C}' = \cup_{d_{\min} \in \mathcal{D}_{\min}} \mathcal{C}'(d_{\min})$ using (37){MCC}
  $(i_{\min}, k_{\min}) \leftarrow \arg\min_{i \in \mathcal{C}'(d_{\min}), k \in S_i} \Delta P_i^k / \overline{P}_i$ such that $P_i + \Delta P_i^k < \overline{P}_i$ {greedy allocation}
  if $i_{\min}$ does not exist **then**
    break{algorithm has failed}
  end if
  $r_{i_{\min}} \leftarrow r_{i_{\min}} + \Delta f$ {1-bit loading}
  $P_{i_{\min}}^{k_{\min}} \leftarrow P_{i_{\min}}^{k_{\min}} + \Delta P_{i_{\min}}^{k_{\min}}$
  $S_j \leftarrow S_j \setminus \{k_{\min}\}, \quad \forall j \in \mathcal{F}_{i_{\min}}$ {avoid interference}
  update $\Delta P_{i_{\min}}^{k_{\min}}$ according to (17)
  calculate $r_{i_{\min}}$ according to (7)
until $\min_d c_{\min}(1, d) \geq \overline{r}$,

allocation:

$$\text{minimize} \quad \sum_{i=1}^{M} P_i, \tag{43a}$$

$$\text{subject to} \quad P_i = \sum_{k \in S_i} P_i^k, \quad P_i^k \geq 0 \tag{43b}$$

$$r_i \leq \Delta f \sum_{k \in S_i} \log_2(1 + \beta P_i^k \rho_i^k), \tag{43c}$$

$$\forall d \in \mathcal{D}:$$

$$\sum_{j \in \mathcal{R}_i} f_{i,j}^d - \sum_{i \in \mathcal{R}_j} f_{j,i}^d = 0, \quad \forall i \neq 1, d \tag{43d}$$

$$\sum_{j \in \mathcal{R}_i} f_{i,j}^d \leq r_i, \quad \forall i, \tag{43e}$$

$$\sum_{j \in \mathcal{R}_1} f_{1,j}^d - \sum_{1 \in \mathcal{R}_j} f_{j,1}^d \geq \overline{r}, \tag{43f}$$

$$f_{i,j}^d \geq 0, \quad \forall i \text{ and } j \in \mathcal{R}_i, \tag{43g}$$

$$\text{variables} \quad \{P_i^k\}, \{r_i\}, \{f_{i,j}\}.$$

A. Two-Stage Algorithm with Max-Min Fair Subcarrier Assignment

The first reference algorithm assigns subcarriers based on a max-min fairness criterion, i.e., it seeks to maximize the minimum number of subcarriers assigned to any node. Let $s_i^k \in \{0, 1\}$ be a variable that indicates if subcarrier k is assigned to node i ($s_i^k = 1$) or not ($s_i^k = 0$), the problem can be stated as a binary program:

$$\text{maximize} \quad t, \tag{44a}$$

$$\text{subject to} \quad t \leq \sum_{k=1}^{K} s_i^k, \quad i \in \mathcal{M}, \tag{44b}$$

$$s_i^k + s_j^k \leq 1, \quad \forall i \in \mathcal{M} \text{ and } j \in \mathcal{R}_i, \tag{44c}$$

$$\text{variables} \quad t \in \mathbb{Z}, \{s_i^k\} \in \{0, 1\}.$$

It turns out that (44) is equivalent to the *Graph Coloring* problem [19] which is NP-complete. Therefore we solve (44) suboptimally with a greedy assignment algorithm, as shown in Alg. 4. We first identify a set $\mathcal{I}$ of nodes that have the least number of assigned subcarriers, then we randomly generate a *maximal independent set* (MIS) in $\mathcal{I}$, which is further randomly enlarged into a MIS in $\mathcal{M}$. An *independent set* (IS, cf. [19]) is a node set in which none of them is interfering with each other. A MIS is an IS that is not a proper subset of any IS. A new subcarrier is assigned to the resulting MIS in $\mathcal{M}$. Note the subroutine $\mathbf{RandMIS}(\mathcal{I}_1, \mathcal{I}_2)$ randomly enlarges an IS from $\mathcal{I}_2 \subset \mathcal{I}_1$ to a MIS in $\mathcal{I}_1$, as shown in Alg. 5.

Alg. 4 Greedy Algorithm for Max-Min Subcarrier Assignment

Require: $s_i^k = 0$

for $k = 1$ to K **do**
  $\mathcal{I} \leftarrow \arg\min_i \sum_{k=1}^{K} s_i^k$ {*nodes with least subcarriers*}
  $\mathcal{I} \leftarrow \mathbf{RandMIS}(\mathcal{I}, \emptyset)$ *find a MIS in* $\mathcal{I}$
  $\mathcal{I} \leftarrow \mathbf{RandMIS}(\mathcal{M}, \mathcal{I})$ {*enlarge it into a MIS in* $\mathcal{M}$}
  $s_i^k \leftarrow = 1, \quad \forall i \in \mathcal{I}$ {*assign them a subcarrier*}
end for

Alg. 5 Subroutine $\mathcal{I} = \mathbf{RandMIS}(\mathcal{I}_1, \mathcal{I}_2)$

Require: $\mathcal{I}_2$ is an IS in $\mathcal{I}_1$

$\mathcal{I} \leftarrow \mathcal{I}_2$
while $\mathcal{I}_1 \setminus \mathcal{I} \neq \emptyset$ **do**
  randomly pick $j \in \mathcal{I}_1 \setminus \mathcal{I}$
  $\mathcal{I}_1 \leftarrow \mathcal{I}_1 \setminus \{j\}$
  if $j \notin \cup_{i \in \mathcal{I}} \mathcal{F}_i$ **then**
    $\mathcal{I} \leftarrow \mathcal{I} \cup \{j\}$ {*if* j *does not interfere with* $\mathcal{I}$, *add it*}
  end if
end while

B. Two-Stage Algorithm with an Interference-Free Assumption

If we relax the interference model, i.e., assume interference-free operation, we would have the subcarrier assignment

$$S_i = \{1, 2, \ldots, K\}, \quad \forall i \in \mathcal{M}. \tag{45}$$

This imaginary assignment would produce strictly the most optimal value for (45), since it allows the largest feasible set for (45). Therefore it serves as a lower bound on performance.

VI. Numerical Results

In this section, we compare the performance of the greedy algorithm with MCC with those of the reference algorithms. We simulate a network with nodes placed in a $5000\text{m} \times 5000\text{m}$ region. Two nodes i and j are considered to be within each other's transmission range if the distance $d_{i,j}$ between them is less than 1000m. We consider $j \in \mathcal{F}_i$ if $\mathcal{R}_i \cap \mathcal{R}_j \neq \emptyset$. This rule is designed to avoid the hidden node problem. M nodes will be randomly placed repetitively until at least $\lceil M/2 \rceil$ nodes are path connected to node 1, the default source node. Among these connected nodes, four nodes other than node

1 are randomly chosen as the destinations for multicast. The target multicast throughput $\overline{r}$ is set to 20kbps. We assume that, for the greedy algorithm, one bit is loaded (1bit/Hz) each time onto a subcarrier (cf. (17)). The path loss model follows the one used in [8]

$$\mathrm{PL} = 128.1 + 37.6 \log_{10} d_{i,j} \ \ (\mathrm{dB}) \tag{46}$$

with $d_{i,j}$ in km. A small-scale fluctuation that is i.i.d. normal with a standard deviation of 8dB is further added to the path loss to produce the channel state (in dB) for $H_{i,j}^k$, $\forall i, j, k$, which is assumed to be static. Our results are obtained by averaging over 40 channel realizations. The subcarrier has a 10kHz bandwidth ($\Delta f = 10$kHz) and the noise density at every receiver is identically -174dBm/Hz. The SNR gap in (1) is chosen to be [12]

$$\beta = -\frac{1.5}{5 \log_2 \mathrm{BER}} \tag{47}$$

with $\mathrm{BER} = 10^{-5}$.

Fig. 2 shows the total transmit power required to achieve $\overline{r} = 20$kbps without individual power constraints as a function of the number of nodes. The number of subcarriers is fixed at 64. It can be observed that with the increasing number of nodes involved in the multicast, the total transmit power becomes larger. However, the greedy algorithm with MCC almost always performs within 1dB from the lower bound and more than 3dB better than the two-stage algorithm with max-min fair subcarrier assignment.

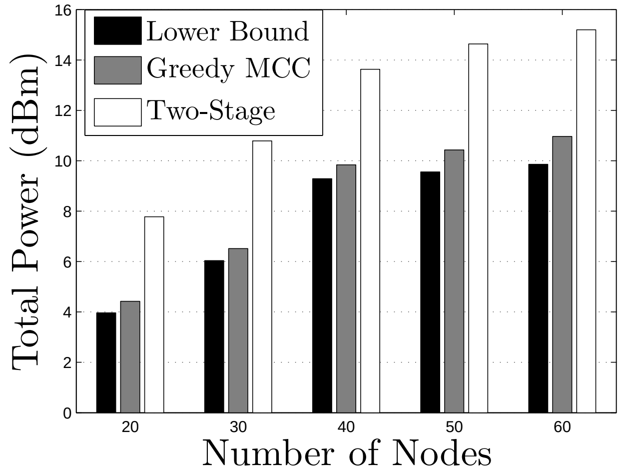

Fig. 2. Comparison in total transmit power of the lower bound, greedy with MCC and two-stage algorithms, with 64 subcarriers and variable number of nodes.

Fig. 3 shows the total transmit power required to achieve $\overline{r} = 20$kbps without individual power constraints as a function of the number of subcarriers. The number of nodes is fixed at 30. It can be observed that with the increasing number of subcarriers (i.e., larger total bandwidth), the total transmit power becomes smaller. However, the greedy algorithm with min-cut chasing almost always performs within 1dB from the lower bound and more than 3dB better than the two-stage algorithm with max-min fair subcarrier assignment.

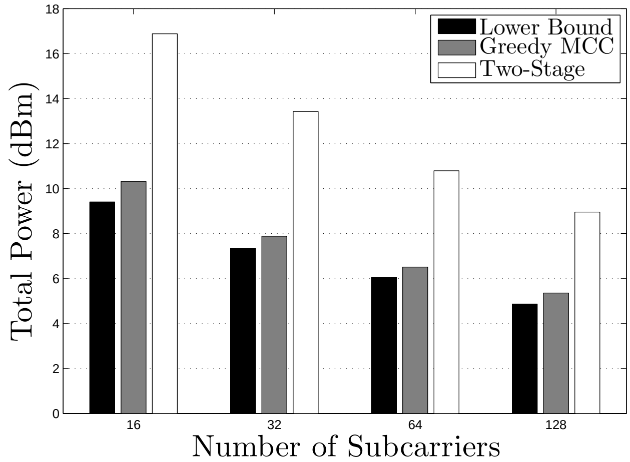

Fig. 3. Comparison in total transmit power of the lower bound, greedy with MCC and two-stage algorithms, with 30 nodes and variable number of subcarriers.

To study the effect of individual power constraints, we consider 36 nodes (one source and three destinations) on a 5000m $\times$ 5000m grid, with individual power constraints as shown in Fig. 4. Specifically, we set the power constraint of node i as

$$\overline{P}_i = 3 - \frac{d_{i,c}}{250} \times 0.3 \ (\mathrm{dBm}). \tag{48}$$

where $d_{i,c}$ is the distance in meters from node i to the center of the grid.

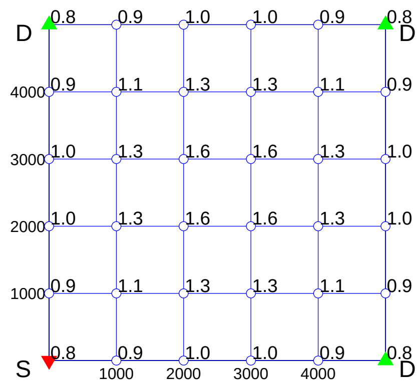

Fig. 4. A grid OFDMA network with 36 nodes (one source (S) & three destinations (D)) and individual power constraints in dBm.

Note that under individual power constraints, the proposed algorithms may not produce a feasible solution. Fig. 5 shows the fraction of trials for which feasible solutions are produced, as a function of $\overline{r}$. It is observed that as $\overline{r}$ increases, the greedy algorithm with MCC maintains a level of feasibility comparable to the interference-free lower bound. Thus it is more robust than the two-stage algorithm with max-min fair subcarrier assignment whose ability to produce a feasible solution declines rapidly. Fig. 6 shows the total transmit power required as a function of $\overline{r}$. We observe that the greedy algorithm with MCC has nearly constant 3dB performance loss compared to the lower bound and is marginally better than the two-stage algorithm in terms of the averaged total transmit power when the algorithms produce a feasible solution.

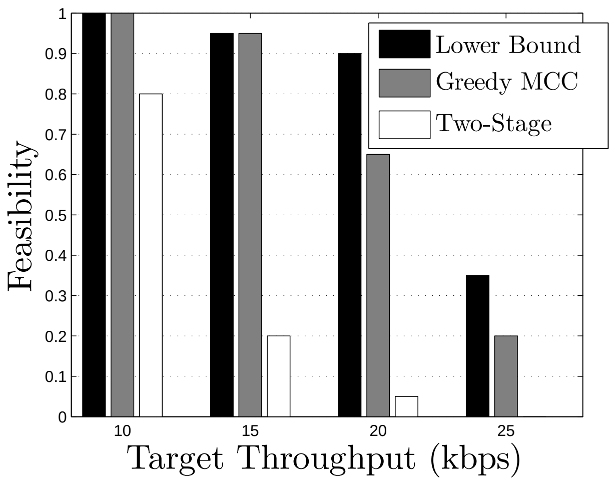

Fig. 5. Comparison in algorithm failure rate on the 36-node grid network.

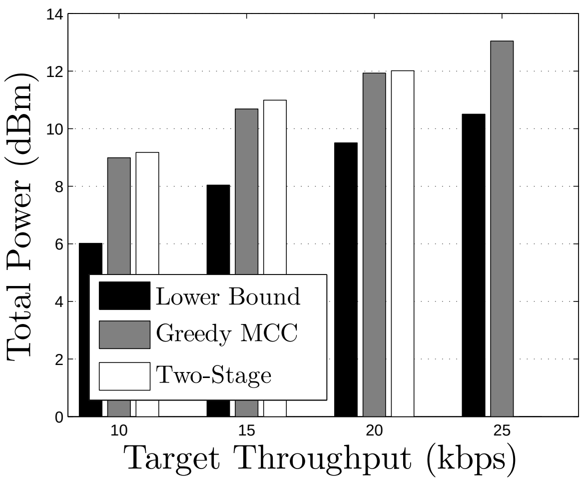

Fig. 6. Comparison in total transmit power of the lower bound, greedy with MCC and two-stage algorithms on a 36-node grid network with variable number of target throughput and 64 subcarriers.

VII. Conclusion and Future Work

In this paper, we described a a greedy resource allocation algorithm to support a multicast in a multihop OFDMA network using random network coding. In particular, we proposed an ancillary node selection procedure to facilitate the greedy iteration, which we refer to as "min-cut chasing." The greedy algorithm with MCC performs nearly optimally when there are no individual power constraints compared to a lower bound based on an interference-free assumption and at least 3dB better than a reference algorithm based on the max-min fair subcarrier assignment. When individual power constraints are imposed, the greedy algorithm with MCC shows robust performance with regard to different throughput requirements. The problem can be extended to cover multiple multicast sessions.

References

[1] R. Ahlswede, N. Cai, S.-Y. Li, and R. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, no. 4, pp. 1204–1216, July 2000.

[2] T. Ho, R. Koetter, M. Médard, D. Karger, and M. Effros, "The benefits of coding over routing in a randomized setting," in *IEEE Int. Symp. Information Theory*, Yokohama, Japan, Jun./Jul. 2003, p. 442.

[3] K. Jain, M. Mahdian, and M. R. Salavatipour, "Packing Steiner trees," in 14^{th} *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2003.

[4] C. Y. Wong, R. S. Cheng, K. B. Letaief, and R. D. Murch, "Multiuser OFDM with adaptive subcarrier, bit, and power allocation," *IEEE J. Select. Areas Comm.*, vol. 17, no. 10, pp. 1747–1758, Oct. 1999.

[5] Y. Zhang and C. Leung, "Distributed resource allocation for multiuser OFDM wireless systems," in *Proc. 3rd ACM workshop on Performance monitoring and measurement of heterogeneous wireless and wired networks*, 2008, pp. 112–116.

[6] D. T. Ngo, C. Tellambura, and H. H. Nguyen, "Efficient resource allocation for OFDMA multicast systems with fairness consideration," in *Radio and Wireless Symposium (RWS'09)*. IEEE, 2009, pp. 392–395.

[7] G. Song and Y. Li, "Cross-layer optimization for OFDM wireless networks-part I: Theoretical framework," *IEEE Trans. Wireless Commun.*, vol. 4, no. 2, pp. 614–624, Mar. 2005.

[8] ——, "Cross-layer optimization for OFDM wireless networks — part II: Algorithm development," *IEEE Trans. Wireless Commun.*, vol. 4, no. 2, pp. 625–634, Mar. 2005.

[9] Y. Xu, X. Wu, J. C. S. Lui, J. C. S. Lui, and X. Wu, "Cross-layer QoS scheduling for layered multicast streaming in OFDMA wireless networks," *Wireless Pers Commun*, no. 51, pp. 565–591, 2009.

[10] Y. Xu, J. C. S. Lui, and D.-M. Chiu, "Analysis and scheduling of practical network coding in OFDMA relay networks," *Computer Networks*, vol. 53, no. 12, pp. 2120–2139, Aug. 2009.

[11] D. S. Lun, N. Ratnakar, M. Médard, R. Koetter, D. R. Karger, T. Ho, E. Ahmed, and F. Zhao, "Minimum-cost multicast over coded packet networks," *IEEE Trans. Inform. Theory*, vol. 52, no. 6, pp. 2608–2623, Jun. 2006.

[12] X. Qiu and K. Chawla, "On the performance of adaptive modulation in cellular systems," *IEEE Trans. Commun.*, vol. 47, no. 6, pp. 884–895, Jun. 1999.

[13] D. S. Lun, M. Medard, R. Koetter, and M. Effros, "On coding for reliable communication over packet networks," in *42nd Annual Allerton Conference on Communication, Control, and Computing*, Sept. 2004.

[14] D. Zhang, N. Mandayam, and S. Parekh, "DEDI: A framework for analyzing rank evolution in random network coding," in *IEEE International Symposium on Information Theory*, Austin, TX, Jun. 2010.

[15] Y. Sagduyu and A. Ephremides, "On joint MAC and network coding in wireless ad hoc networks," *IEEE Trans. Inform. Theory*, vol. 53, no. 10, pp. 3697–3713, Oct. 2007.

[16] L. Wan and J. Luo, "Wireless multicasting via iterative optimization," in *Proc. International Symposium on Information Theory (ISIT)*, Jun. 2010, pp. 2333–2337.

[17] S. Martello and P. Toth, *Knapsack Problems: Algorithms and Computer Implementations*, ser. Wiley-Interscience series in discrete mathematics and optimization. Chichester, England: John Wiley & Sons Ltd., 1990.

[18] D. P. Bertsekas, *Network Optimization: Continuous and Discrete Models*. Belmont, Massachusetts: Athena Scientific, 1998.

[19] M. Kubale, *Graph Colorings*, ser. Contemporary Mathematics. American Mathematical Society, 2004.