Analyzing Multiple Flows in a Wireless Network with Differential Equations and Differential Inclusions

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***Abstract*—A framework based on differential equations and differential inclusions (DEDI) for analyzing the throughput of random network coding in a wireless network has been proposed in [1]. In this paper, we extend this framework to further address the throughput when multiple multicast flows are simultaneously launched using an intersession coding scheme. The handiness, accuracy and power of the DEDI framework are demonstrated through its use in theoretical analysis and simulations.**

I. Introduction

Analyzing information flows with network coding was pioneered by Ahlswede *et al.* in [2], where the throughput of a single multicast flow in a wired network using deterministic network coding was analyzed. They showed for the first time that with proper coding, the throughput of a multicast is tightly upper bounded by the throughput of individual unicasts. Similar results for random network coding (RNC) were reported in [3], [4] using queueing theoretic methods. Since then, analyzing throughput of information flows in a wired or wireless network has been extensively studied and many methods have been proposed. However, these methods, mostly based on information theory or queueing theory, are not readily amenable to use in networking practice, due to the lack of either generality or simplicity. In our recent work [1], we have presented a new framework (DEDI) based on differential equations (DE) and differential inclusions (DI) for analyzing the dynamics of network coding, namely the evolution of rank in a wireless network with RNC. Specifically, this framework can be applied to study network information flows and throughput with RNC via a dynamical systems approach. We used this approach to reprove one of the fundamental theorems of a single network flow in [1], first proved in [3], thereby demonstrating the versatility of the DEDI framework in theory and design. In this paper, we will use the DEDI framework to further extend the results from a single flow to multiple flows starting simultaneously. We will characterize the throughput, and demonstrate the use of the DEDI framework to network practitioners.

In what follows, Section II provides a brief review of wireless network hypergraph and RNC basics; Section III presents an abbreviated description of the setup of DEDI from [1]; Section IV discusses its application in characterizing the throughput of multiple RNC flows, proving a few useful theorems; Section V presents simulations to illustrate the accuracy of DEDI. We conclude in Section VI.

II. Hypergraph and Random Network Coding

A generic wireless network is modeled as a hypergraph $G = (\mathcal{N}, \mathcal{E})$ consisting of N nodes $\mathcal{N} = \{1, 2, \ldots, N\}$ and hyperarcs $\mathcal{E} = \{(i, \mathcal{K}) | i \in \mathcal{N}, \mathcal{K} \subset \mathcal{N}\}$. Each hyperarc captures the fact that wireless transmission is naturally broadcasting. This idea is shown in Fig. 1 where the hypergraph of a four-node network is shown. It can be conveniently represented with arrows to indicate reachability. One should not, however, confuse the arrow representation with the digraph of a wired network. Assume some underlying MAC is operating in its steady state such that each node i is transmitting at λ_i packets per second. We say that a packet is successfully received by a set $\mathcal{K}$ of nodes if the packet is successfully received by at least one node in $\mathcal{K}$, which happens with a probability $P_{i,\mathcal{K}}$. We define the transmit rate $z_{i,\mathcal{K}}$ for $(i, \mathcal{K})$ (i.e., from i to $\mathcal{K}$) as

$$z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}. \tag{1}$$

For $\mathcal{K} \subset \mathcal{T} \subset \mathcal{N}$, we must have

$$z_{i,\mathcal{K}} \leq z_{i,\mathcal{T}} \tag{2}$$

because $P_{i,\mathcal{K}} \leq P_{i,\mathcal{T}}$. Suppose $\mathcal{S}, \mathcal{K} \subset \mathcal{N}$ and $\mathcal{S} \cap \mathcal{K} = \emptyset$. Define a cut for the pair $(\mathcal{S}, \mathcal{K})$ as a set $\mathcal{T}$ satisfying $\mathcal{K} \subset \mathcal{T} \subset \mathcal{S}^c$. Let $C(\mathcal{S}, \mathcal{K})$ denote the collection of all cuts for $(\mathcal{S}, \mathcal{K})$. The size of $\mathcal{T}$ is defined as $c(\mathcal{T}) = \sum_{i \in \mathcal{T}^c} z_{i,\mathcal{T}}$. The min cut $\mathcal{T}_{\min}$ for $(\mathcal{S}, \mathcal{K})$, whose size is denoted as $c_{\min}(\mathcal{S}, \mathcal{K})$ is a cut satisfying

$$c(\mathcal{T}_{\min}) = \min_{\mathcal{T}' \in C(\mathcal{S}, \mathcal{K})} c(\mathcal{T}'). \tag{3}$$

We denote $T_{\min}$ as $C_{\min}(\mathcal{S}, \mathcal{K})$. Finally, we say G is connected if, for any $\emptyset \neq \mathcal{T} \subsetneq \mathcal{N}$, $c(\mathcal{T}) > 0$.

For the basic operation of RNC, we refer the unfamiliar readers to the original paper [4] and we only make a few notions here for the development of DEDI. The number of innovative packets at node i (including the source packets if i is a source node) is denoted as V_i. Note V_i is also the number of linearly independent global coefficient vectors [5]. Let S_i be the vector space spanned by those global coefficient vectors

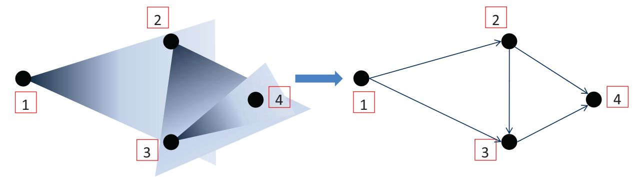

Fig. 1. Hypergraph model of a wireless network of four nodes and its arrow-dot representation.

over the underlying field GF(q), then $V_i = \dim S_i$, which we call the *rank* of node i. S_i and V_i are time dependent as the coded transmissions continues. Further, for any set $\mathcal{K} \subset \mathcal{N}$, define

$$S_{\mathcal{K}} = \sum_{i\in\mathcal{K}} S_i, \quad V_{\mathcal{K}} = \dim S_{\mathcal{K}}, \tag{4}$$

and call $V_{\mathcal{K}}$ the rank of $\mathcal{K}$. The question we are interested in answering is how the rank $V_{\mathcal{K}}$ or V_i evolves over time.

III. The DEDI Framework

A. Rank Evolution Modeled with DE

Under the fluid approximation [6] $V_i(t)$ can be closely represented by its mean $E[V_i(t)]$. The DEDI framework models $E[V_i(t)]$ with a set of DE's with which all kinds of analysis can be carried out easily. Thus in what follows, we drop the notation $E[\cdot]$ and use $V_i(t)$ or $V_{\mathcal{K}}(t)$ to denote their average values, respectively, as functions of t. Since V_i has been turned from a discrete incremental process into a deterministic smooth function under the fluid approximation, we may consider the differential of $V_{\mathcal{K}}$. For the moment, assume all nodes in the network participate in carrying the flow with RNC, i.e., they constantly form a coded packet and broadcast it out as scheduled by their MAC's. During a diminishing time interval Δt, an average of

$$\Delta t \sum_{j\notin\mathcal{K}} \lambda_j \ll 1$$

(fractional) packet originating from some node of $\mathcal{K}^c$. If this packet comes from $i \notin \mathcal{K}$ (with probability $\lambda_i / \sum_{j\notin\mathcal{K}} \lambda_j$), it is successfully received by $\mathcal{K}$ with probability $P_{i,\mathcal{K}}$ by definition. This packet then increases $V_{\mathcal{K}}$ by 1 if and only if its associated global coefficient vector comes from $S_i \setminus (S_i \cap S_{\mathcal{K}})$. Since

$$|S_i \cap S_{\mathcal{K}}| = q^{\dim S_i \cap S_{\mathcal{K}}} = q^{V_i + V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}, \tag{5}$$

$$\text{and} \quad |S_i| = q^{\dim S_i} = q^{V_i}, \tag{6}$$

it happens with probability

$$(|S_i| - |S_i \cap S_{\mathcal{K}}|)/|S_i| = 1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}. \tag{7}$$

Consequently in Δt, $V_{\mathcal{K}}$ is incremented by

$$\left(\Delta t \sum_{j\notin\mathcal{K}} \lambda_j\right) \sum_{i\notin\mathcal{K}} \frac{\lambda_i}{\sum_{j\notin\mathcal{K}} \lambda_j} P_{i,\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}})$$
$$= \Delta t \sum_{i\notin\mathcal{K}} \lambda_i P_{i,\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}), \tag{8}$$

i.e.,

$$V_{\mathcal{K}}(t+\Delta t) - V_{\mathcal{K}}(t) = \Delta t \sum_{i\notin\mathcal{K}} \lambda_i P_{i,\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}).$$

Using $z_{i\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}$ and equation (5), we write $\forall \mathcal{K} \subset \mathcal{N}$

$$\dot{V}_{\mathcal{K}} = \lim_{\Delta t \to 0} \frac{V_{\mathcal{K}}(t+\Delta t) - V_{\mathcal{K}}(t)}{\Delta t} = \sum_{i\notin\mathcal{K}} z_{i\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}). \tag{9}$$

The nonlinear DE in (9) forms the basis for the DEDI framework. It actually stands for a system of $2^N - 1$ equations, each for an nonempty $\mathcal{K} \subset \mathcal{N}$. They collectively give a complete description of rank evolution in the system. Note $V_{\mathcal{K}}$ is solely determined by $\{V_{\mathcal{K}\cup\{i\}}\}_{i\notin\mathcal{K}}$. This dependency can be explored to arrange (9) into a partial order "$\lesssim$" such that $V_{\mathcal{K}} \lesssim V_{\mathcal{L}}$ if and only if $\mathcal{K} \subset \mathcal{L}$. This partial order can be pictorially represented as a layered structure, for which an example is shown in Fig. 2 for $|\mathcal{N}| = 3$. To determined a quantity on any particular layer, one only needs to know the the quantities on the layer immediately above indicated by arrows.

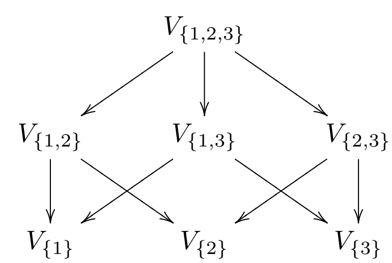

Fig. 2. Layered structure for the rank evolution of a 3-node network

Theoretically, with appropriate boundary condition, (9) can be solved. The instantaneous throughput is then obtained as $\dot{V}_{\mathcal{K}}$ or $\dot{V}_i$. For example, assuming node 1 is the unique source with m packets to deliver, the boundary conditions (B.C.) for this systems of DE's are

$$V_{\mathcal{K}}(0) = \begin{cases} m, & 1 \in \mathcal{K}, \\ 0, & \text{o.w.} \end{cases} \tag{10}$$

If only part of the nodes, say $\mathcal{L} \subset \mathcal{N}$, participate in carrying the flow, (9) still holds, except that we should replace $\mathcal{K}$ with $\mathcal{K} \cap \mathcal{L}$ and the top layer in the layered structure consists of $V_{\mathcal{L}}$ alone.

The size q (usually an integral power of 2) of the underlying field GF(q) allows simplification of (9) by making the following approximation

$$q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}} = \begin{cases} 0, & V_{\mathcal{K}} < V_{\mathcal{K}\cup\{i\}}, \\ 1, & V_{\mathcal{K}} = V_{\mathcal{K}\cup\{i\}}, \end{cases} \tag{11}$$

Then we may rewrite (9) as

$$\dot{V}_{\mathcal{K}} = \sum_{i\notin\mathcal{K}} z_{i\mathcal{K}} (V_{\mathcal{K}\cup\{i\}} \ominus V_{\mathcal{K}}), \quad \forall \mathcal{K} \subset \mathcal{N} \tag{12}$$

with the same boundary conditions as in (10). The binary operation $\ominus$ is defined as

$$x \ominus y = \begin{cases} 1, & x > y, \\ 0, & \text{o.w.} \end{cases} \tag{13}$$

B. Rank Evolution Modeled with DI

While (12) is appropriate for numerical evaluation, it is difficult for analysis due to the discontinuous right-hand side. Instead of going back to (9), which is smooth though neither suitable for numerical evaluation nor analysis, we modify the right-hand sides to incorporate semicontinuity. Specifically, we define an upper semicontinuous function $\mathrm{Sgn}^+ : \mathbb{R} \to 2^{\mathbb{R}}$

$$\mathrm{Sgn}^+(x) = \begin{cases} \{0\}, & x < 0 \\ [0, 1], & x = 0 \\ \{1\}, & x > 0 \end{cases} \tag{14}$$

to replace the "$\ominus$" operation:

$$\dot{V}_{\mathcal{K}} \in \sum_{i \notin \mathcal{K}} z_{i\mathcal{K}} \mathrm{Sgn}^+(V_{\mathcal{K} \cup \{i\}} - V_{\mathcal{K}}), \quad \forall \mathcal{K} \subset \mathcal{N}. \tag{15}$$

The same boundary condition in (10) still holds. To be compatible with (12), when $\mathcal{K} = \mathcal{N}$, we define the right-hand side of (15) to be $\{0\}$ instead of $\emptyset$. In mathematical literature, (15) plus (10) is called a system of differential inclusions (DI) [7], which is an important extension to regular DE's to allow jump discontinuities. For this paper, solutions to the DI's as in (15) are defined to be all the continuous functions that satisfy (15) and are differentiable almost everywhere. Under mild assumptions, the DI's admit a unique solution that also satisfies the associated DE's with discontinuities.

IV. Analyzing Multiple Flows with DEDI

In [2], the throughput for deterministic network coding in a wired network was established. The result states that, for a single multicast flow with a single source node, a multicast rate of R can be achieved if a unicast at rate R can be achieved for each destination separately. In fact, since it is well known that the unicast throughput is determined by the min cut between the source and the destination, the result essentially states that the multicast throughput is determined by the smallest min cut between the source and each destination. This result is astonishing in that one cannot achieve the throughput simply by routing in a store-and-forward fashion. Similar results for RNC in a wired or wireless network were established in [3] with the hypergraph model and the min cut concept defined as in (3). In [1], the authors proved a slightly stronger statement by solving (15). They showed that for any $\mathcal{K} \subset \mathcal{N}$ that does not include the source (assumed as node 1), its rank increases at the min cut size, i.e., $\dot{V}_{\mathcal{K}} = c_{\min}(1, \mathcal{K})$. The original statement in [3] is then recovered by specializing $\mathcal{K}$ to a single node i.

It would be interesting to see how DEDI could be applied to the more complicated scenario of multiple flows. This can be done, as always, by solving (15) that describes this scenario. In general, suppose we have a wireless network $G = (\mathcal{N}, \mathcal{E})$ and J independent multicast flows and flow j originates from a set of source nodes

$$\mathcal{S}_j = \{s_{j,1}, s_{j,2}, \ldots, s_{j,n_j}\}, \quad j = 1, 2, \ldots, J, \tag{16}$$

where each node in $\mathcal{S}_j$ contains the same set of m_j packets to be delivered to the rest of the network or part of it. Note it is possible that a node serves more than one multicast flows and it contains as many sets of packets. To identify the source for any nonempty $\mathcal{K} \subset \mathcal{N}$, define

$$\mathrm{Src}(\mathcal{K}) = \{j | \mathcal{S}_j \cap \mathcal{K} = \emptyset, j = 1, 2, \ldots, J\}. \tag{17}$$

For the coding scheme, we let each node generate a coded packet by randomly linearly mixing all the packets it holds, regardless which multicast flow these packets belong to. Suppose all the multicast flows start synchronously from time 0. This scenario is captured by the following system of DI's:

$$\begin{aligned} &\dot{V}_{\mathcal{K}} \in \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\} \cup \mathcal{K}} - V_{\mathcal{K}}), \\ &\text{B.C.} \quad V_{\mathcal{K}}(0) = \sum_{\substack{1 \le j \le J \\ j \notin \mathrm{Src}(\mathcal{K})}} m_j. \end{aligned} \tag{18}$$

We now state the following theorem which provides an explicit solution to (18):

Theorem 1: The solution to (18) is given recursively as

$$V_{\mathcal{K}}(t) = \min\{V_{\mathcal{K}}(0) + c(\mathcal{K})t, \min_{\ell \notin \mathcal{K}}\{V_{\{\ell\} \cup \mathcal{K}}(t)\}\}, \tag{19}$$

$$= \min\{V_{\mathcal{K}}(0) + c(\mathcal{K})t, \min_{\mathcal{K}' \supset \mathcal{K}}\{V_{\mathcal{K}'}(t)\}\}, \tag{20}$$

$$= \min_{\mathcal{K}' \supset \mathcal{K}}\{V'_{\mathcal{K}}(0) + c(\mathcal{K}')t\} \tag{21}$$

and

$$V_{\mathcal{N}}(t) = \sum_{j=1}^{J} m_j. \tag{22}$$

Besides, for each $\mathcal{K}$, there is a sequence

$$0 = t_0 < t_1 < \cdots < t_{n_{\mathcal{K}}-1} < t_{n_{\mathcal{K}}} = \infty \tag{23}$$

such that over $[t_p, t_{p+1})$, $p = 0, 1, \ldots, n_{\mathcal{K}} - 1$, $V_{\mathcal{K}}$ is affine:

$$V_{\mathcal{K}}(t) = V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \quad t \in [t_p, t_{p+1}), \tag{24}$$

where $\mathcal{K}'$ satisfies

$$\mathcal{K}' = C_{\min}(\cup_{j \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_j, \mathcal{K}). \tag{25}$$

We need a few preliminaries for the proof of Theorem 1. We begin with Lemma 1 which gives a solution to (18) on an interval.

Lemma 1: Suppose $V_{\mathcal{K}}(t_1)$ is known, (18) has a solution on $[t_1, t_2)$ given as

$$V_{\mathcal{K}}(t) = V_{\mathcal{K}}(t_1) + z(t - t_1) \tag{26}$$

if there is a set of nodes $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{K} = \emptyset$ and $z \ge 0$ satisfying

1) $V_{\{i\} \cup \mathcal{K}}(t) = V_{\mathcal{K}}(t_1) + z(t - t_1)$, $\forall t \in [t_1, t_2)$, $\forall i \in \mathcal{P}$;
2) $V_{\{j\} \cup \mathcal{K}} \ge V_{\mathcal{K}}(t_1) + z(t - t_1)$, $\forall t \in [t_1, t_2)$, $\forall j \notin \mathcal{K} \cup \mathcal{P}$;

3) $\sum_{j\notin\mathcal{K}\cup\mathcal{P}} z_{j,\mathcal{K}} \leq z \leq \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}}$.

Proof: Suppose this is not true, there is $t'' \in [t_1, t_2)$ such that

$$V_{\mathcal{K}}(t'') \neq V_{\mathcal{K}}(t_1) + z(t - t_1).$$

Let

$$t' = \sup_{t_1 \leq t \leq t''} \{V_{\mathcal{K}}(t) = V_{\mathcal{K}}(t_1) + z(t - t_1)\}, \tag{27}$$

then t' exists, $t_1 \leq t' < t'' < t_2$ and $V_{\mathcal{K}}(t) \neq V_{\mathcal{K}}(t_1)+z(t-t_1)$ $\forall t \in (t', t'']$ by definition. Because $V_{\mathcal{K}}(t)$ is continuous, it is either

$$V_{\mathcal{K}}(t) > V_{\mathcal{K}}(t_1) + z(t - t_1), \quad \forall t \in (t', t''], \tag{28}$$

or

$$V_{\mathcal{K}}(t) < V_{\mathcal{K}}(t_1) + z(t - t_1), \quad \forall t \in (t', t'']. \tag{29}$$

If (28) holds, by assumption 1, $V_{\{i\}\cup\mathcal{K}}(t) < V_{\mathcal{K}}(t)$, $\forall t \in (t', t'']$, $\forall i \in \mathcal{P}$. So, with assumption 2,

$$\dot{V}_{\mathcal{K}}(t) = \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}) \leq \sum_{j\notin\mathcal{K}\cup\mathcal{P}} z_{j,\mathcal{K}}, \tag{30}$$

thus

$$\begin{aligned}
V_{\mathcal{K}}(t'') &= V_{\mathcal{K}}(t') + \int_{t'}^{t''} \dot{V}_{\mathcal{K}}(t)dt \\
&= V_{\mathcal{K}}(t_1) + z(t' - t_1) + \int_{t'}^{t''} \dot{V}_{\mathcal{K}}(t)dt \\
&\leq V_{\mathcal{K}}(t_1) + (t' - t_1) + \sum_{j\notin\mathcal{K}\cup\mathcal{P}} z_{j,\mathcal{K}}(t'' - t') \\
&\leq V_{\mathcal{K}}(t_1) + z(t' - t_1) + z(t'' - t') \quad \text{(assumption 3)} \\
&= V_{\mathcal{K}}(t_1) + z(t'' - t_1),
\end{aligned} \tag{31}$$

which is a contradiction to (28). If (29) holds, by assumption 1 and 2,

$$\dot{V}_{\mathcal{K}} = \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}) = \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}}. \tag{32}$$

Then by assumption 3,

$$\begin{aligned}
V_{\mathcal{K}}(t'') &= V_{\mathcal{K}}(t_1) + z(t' - t_1) + \int_{t'}^{t''} \dot{V}_{\mathcal{K}}(t)dt \\
&= V_{\mathcal{K}}(t_1) + z(t' - t_1) + \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}}(t'' - t') \\
&\geq V_{\mathcal{K}}(t_1) + z(t' - t_1) + z(t'' - t') \quad \text{(assumption 3)} \\
&= V_{\mathcal{K}}(t_1) + z(t'' - t_1),
\end{aligned} \tag{33}$$

which is a contradiction to (29). ■

Now we can give

Proof to Theorem 1: We prove this via induction on $|\mathcal{N} \setminus \mathcal{K}|$. When $|\mathcal{N} \setminus \mathcal{K}| = 0$, $\mathcal{K} = \mathcal{N}$, $V_{\mathcal{N}}(0) = \sum_{j=1}^{J} m_j$, $\forall t \geq 0$. (19)–(21) are trivially true. Assume it is true when $|\mathcal{N} \setminus \mathcal{K}| \leq k-1$, we prove it is also true for $|\mathcal{N} \setminus \mathcal{K}| = k$. Let

$$U_{\mathcal{K}}(t) = \min_{\mathcal{K}'\supset\mathcal{K}} \{V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \tag{34}$$

then $U_{\mathcal{K}}(t)$ is piecewise linear (since it is the minimum of a finitely many affine functions) and there is a sequence

$$0 = t_0 < t_1 < \cdots < t_{n_{\mathcal{K}}-1} < t_{n_{\mathcal{K}}} = \infty \tag{35}$$

such that for each $p = 0, 1, \ldots, n_{\mathcal{K}} - 1$,

$$U_{\mathcal{K}}(t) = V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \quad t \in [t_p, t_{p+1}), \tag{36}$$

for some $\mathcal{K}'$. We claim $\mathcal{K}' = C_{\min}(\cup_{j\in\mathrm{Src}(\mathcal{K}')}\mathcal{S}_j, \mathcal{K})$. Otherwise, let $\mathcal{K}'' = C_{\min}(\cup_{j\in\mathrm{Src}(\mathcal{K}')}\mathcal{S}_j, \mathcal{K})$ but $\mathcal{K}'' \neq \mathcal{K}$. By definition of min cut for the hypergraph model, we have

$$c(\mathcal{K}'') < c(\mathcal{K}'). \tag{37}$$

Since $(\cup_{j\in\mathrm{Src}(\mathcal{K}')}\mathcal{S}_j) \cap \mathcal{K}'' = \emptyset$, $\mathrm{Src}(\mathcal{K}') \subset \mathrm{Src}(\mathcal{K}'')$, so

$$V_{\mathcal{K}''}(0) \leq V_{\mathcal{K}'}(0). \tag{38}$$

Therefore $\forall t \in (t_p, t_{p+1})$,

$$V_{\mathcal{K}''}(0) + c(\mathcal{K}'')t < V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \tag{39}$$

which is a contradiction to (34).

We want to show that $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \in [t_p, t_{p+1})$, using Lemma 1, which amounts to checking three conditions. Let $\mathcal{P} = \mathcal{K}' \setminus \mathcal{K}$, $z = \sum_{i\notin\mathcal{K}'} z_{i,\mathcal{K}'} = c(\mathcal{K}')$. First note

$$V_{\{i\}\cup\mathcal{K}}(t) = U_{\mathcal{K}}(t), \quad \forall i \in \mathcal{P}, \forall t \in [t_p, t_{p+1}). \tag{40}$$

This is because, on one hand, $V_{\{i\}\cup\mathcal{K}}(t) \geq U_{\mathcal{K}}(t)$ by (34), while on the other hand, by induction assumption ($|\mathcal{N} \setminus (\{i\} \cup \mathcal{K})| = k-1$)

$$\begin{aligned}
V_{\{i\}\cup\mathcal{K}}(t) &= \min_{(\{i\}\cup\mathcal{K})\subset\mathcal{K}''} \{V_{\mathcal{K}''}(0) + c(\mathcal{K}'')t\} \\
&\leq V_{\mathcal{K}'}(0) + c(\mathcal{K}')t \quad (\text{since } \{i\} \cup \mathcal{K} \subset \mathcal{K}') \\
&= U_{\mathcal{K}}(t).
\end{aligned} \tag{41}$$

Meanwhile, by (34) we have

$$V_{\{j\}\cup\mathcal{K}}(t) \geq U_{\mathcal{K}}(t), \quad \forall t \in [t_p, t_{p+1}), \forall j \notin \mathcal{K} \cup \mathcal{P}. \tag{42}$$

Because $\mathcal{K}' = C_{\min}(\cup_{j\in\mathrm{Src}(\mathcal{K}')}\mathcal{S}_j, \mathcal{K})$, $z \leq \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}}$. Because $\mathcal{K} \subset \mathcal{K}'$,

$$\sum_{i\notin\mathcal{K}'} z_{i,\mathcal{K}} \leq \sum_{i\notin\mathcal{K}'} z_{i,\mathcal{K}'} = z. \tag{43}$$

Thus assumption 3 of Lemma 1 is checked for all t. We then check assumption 1 and 2 piecewise. When $p = 0$,

$$U_{\mathcal{K}}(t_0) = U_{\mathcal{K}}(0) = \min_{\mathcal{K}'\supset\mathcal{K}} \{V_{\mathcal{K}'}(0)\} = V_{\mathcal{K}}(0) \tag{44}$$

because for any $\mathcal{K}' \supset \mathcal{K}$,

$$\begin{aligned}
V_{\mathcal{K}'}(0) &= \sum_{j=1}^{J} I(\mathcal{S}_j \cap \mathcal{K}' \neq \emptyset)m_j \\
&\geq \sum_{j=1}^{J} I(\mathcal{S}_j \cap \mathcal{K} \neq \emptyset)m_j = V_{\mathcal{K}}(0).
\end{aligned} \tag{45}$$

From (40) and (45),

$$\begin{aligned}
V_{\{i\}\cup\mathcal{K}}(t) &= V_{\mathcal{K}'}(0) + c(\mathcal{K}')t \\
&= V_{\mathcal{K}}(0) + c(\mathcal{K}')t, \quad \forall t \in [t_0, t_1), \forall i \in \mathcal{P},
\end{aligned} \tag{46}$$

Hence assumption 1 is checked for $[t_0, t_1)$. From (42) and (45),

$$V_{\{j\}\cup\mathcal{K}}(t) \geq V_{\mathcal{K}'}(0) + c(\mathcal{K}')t = V_{\mathcal{K}}(0) + c(\mathcal{K}')t, \forall t \in [t_0, t_1), \forall j \notin \mathcal{K} \cup \mathcal{P}. \quad (47)$$

Hence assumption 2 is checked for $[t_0, t_1)$. Therefore $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \in [t_0, t_1)$. But this in turn implies that $V_{\mathcal{K}}(t_1) = U_{\mathcal{K}}(t_1)$ by continuity (cf. the definition of solution to DI in Section III-B), which implies that assumption 1 and 2 are also checked for $[t_1, t_2)$ (same argument as for $[t_0, t_1)$). Therefore $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \in [t_1, t_2)$. Repeat this argument $n_{\mathcal{K}}$ times, we conclude that $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \geq 0$. This shows the validity of (21) for $|\mathcal{N} \setminus \mathcal{K}| = k$.

To complete the induction step, we need to show the equivalence between (19)–(21). First note (21) is simply the expansion of (20) by induction. For the equivalence of (19) and (20), we have by induction

$$(19) = \min\{V_{\mathcal{K}}(0) + c(\mathcal{K})t, \min_{\ell \notin \mathcal{K}}\{V_{\{\ell\}\cup\mathcal{K}}(0) + c(\{\ell\}\cup\mathcal{K})t, \min_{\mathcal{K}'' \supset \{\ell\}\cup\mathcal{K}}\{V_{\mathcal{K}''}(t)\}\} = (21) = (20). \quad (48)$$

■

Essentially, Theorem 1 (cf. (21)) states that $V_{\mathcal{K}}(t)$ is the min-envelop of $2^{|\mathcal{N}\setminus\mathcal{K}|}$ affine functions corresponding to so many subsets of nodes that contain $\mathcal{K}$. The partial order "$\lesssim$" illustrated by the layered structure also implies the usual linear order "$\leq$", i.e.,

$$\mathcal{K} \lesssim \mathcal{K}' \Rightarrow V_{\mathcal{K}}(t) \leq V'_{\mathcal{K}}(t), \quad \forall t \geq 0. \quad (49)$$

Therefore it is always true

$$V_{\mathcal{K}} \leq V_{\mathcal{N}} = \sum_{j=1}^{J} m_j. \quad (50)$$

A stronger statement than (50) can be made when G is connected, i.e.,

Corollary 1: If $G = (\mathcal{N}, \mathcal{E})$ is connected, then $\forall \mathcal{K} \neq \emptyset$

$$V_{\mathcal{K}}(t) = \sum_{j=1}^{J} m_j, \quad t \in [t_{n_{\mathcal{K}}-1}, \infty). \quad (51)$$

Proof: Because G is connected, $\forall \mathcal{K} \subset \mathcal{K}' \subsetneq \mathcal{N}$, $c(\mathcal{K}') > 0$. Therefore when t is sufficiently large,

$$V'_{\mathcal{K}}(0) + c(\mathcal{K}')t > \sum_{j=1}^{J} m_j = V_{\mathcal{N}}(t).$$

Hence we have the conclusion from (21) of Theorem 1. ■

Corollary 1 implies that with the RNC scheme for multiple flows as described here, a node may have to wait until its rank reaches $\sum_{j=1}^{J} m_j$ to start decoding. This time is denoted as $T_{\mathcal{K}}^{\text{total}}$. Though there could be fairly large decoding delay for nodes only interested in one or few sessions, the intersession coding is optimal in the sense of min cut bound. Apply (24) in Theorem 1 to $[t_{n_{\mathcal{K}}-1}, t_{n_{\mathcal{K}}})$, it is clear that $T_{\mathcal{K}}^{\text{total}}$ is determined by one of the min cut bounds that $\mathcal{K}$ has to respect. The min cut that determines the finish time can be regarded as the worst bottleneck for $\mathcal{K}$. More precisely, we have

Theorem 2: If $G = (\mathcal{N}, \mathcal{E})$ is connected, then

$$T_{\mathcal{K}}^{\text{total}} = \max_{S \subset \text{Src}(\mathcal{K})}\{\sum_{j \in S} m_j / c_{\min}(\cup_{i \in S}\mathcal{S}_i, \mathcal{K})\}. \quad (52)$$

Proof: Clearly $T_{\mathcal{K}}^{\text{total}} = t_{n_{\mathcal{K}}-1}$. By Theorem 1, there is $\mathcal{K}' \supset \mathcal{K}$ such that $\forall t \in [t_{n_{\mathcal{K}}-2}, t_{n_{\mathcal{K}}-1})$,

$$V_{\mathcal{K}}(t) = V_{\mathcal{K}'}(0) + c(\mathcal{K}')t = \sum_{j \notin \text{Src}(\mathcal{K}')} m_j + c_{\min}(\cup_{i \in \text{Src}(\mathcal{K}')}\mathcal{S}_i, \mathcal{K})t, \quad (53)$$

and by setting $V_{\mathcal{K}}(t_{n_{\mathcal{K}}-1}) = \sum_{j=1}^{J} m_j$, we get

$$\begin{aligned} T_{\mathcal{K}}^{\text{total}} &= t_{n_{\mathcal{K}}-1} \\ &= \left(\sum_{j=1}^{J} m_j - \sum_{\substack{1 \leq j' \leq J \\ j' \notin \text{Src}(\mathcal{K}')}} m_{j'}\right) / c_{\min}(\cup_{i \in \text{Src}(\mathcal{K}')}\mathcal{S}_i, \mathcal{K}) \\ &= \sum_{j \in \text{Src}(\mathcal{K}')} m_j / c_{\min}(\cup_{i \in \text{Src}(\mathcal{K}')}\mathcal{S}_i, \mathcal{K}) \\ &\leq \max_{S \subset \text{Src}(\mathcal{K})}\{\sum_{j \in S} m_j / c_{\min}(\cup_{i \in S}\mathcal{S}_i, \mathcal{K})\}, \end{aligned} \quad (54)$$

where the last inequality holds because $\mathcal{K}' \supset \mathcal{K}$, hence $\text{Src}(\mathcal{K}') \subset \text{Src}(\mathcal{K})$. However, if there is $S' \subset \text{Src}(\mathcal{K})$, such that

$$\sum_{j \in S'} m_j / c_{\min}(\cup_{i \in S'}\mathcal{S}_i, \mathcal{K}) > T_{\mathcal{K}}^{\text{total}}, \quad (55)$$

let $\mathcal{K}'' = C_{\min}(\cup_{i \in S'}\mathcal{S}_i, \mathcal{K})$, then we have

$$V_{\mathcal{K}''}(0) \leq \sum_{\substack{1 \leq j'' \leq J \\ j'' \notin S'}} m_{j''} \quad (56)$$

because $S' \subset \text{Src}(\mathcal{K}'')$, and

$$\begin{aligned} V_{\mathcal{K}''}(0) + c(\mathcal{K}'')T_{\mathcal{K}}^{\text{total}} &= V_{\mathcal{K}''}(0) + c_{\min}(\cup_{i \in S'}\mathcal{S}_i, \mathcal{K})T_{\mathcal{K}}^{\text{total}} \\ &< \sum_{\substack{1 \leq j'' \leq J \\ j'' \notin S'}} m_{j''} + \sum_{j \in S'} m_j = \sum_{j=1}^{J} m_j = V_{\mathcal{K}}(T_{\mathcal{K}}^{\text{total}}), \end{aligned} \quad (57)$$

which contradicts (49). So

$$T_{\mathcal{K}}^{\text{total}} \geq \max_{S \subset \text{Src}(\mathcal{K})}\{\sum_{j \in S} m_j / c_{\min}(\cup_{i \in S}\mathcal{S}_i, \mathcal{K})\}. \quad (58)$$

Combine (54) with (58), we get (52). ■

With Theorem 2, we can finally characterize the average throughput of multiple flows with RNC. For any set $\mathcal{K}$ subscribing to the same set of flows $F_{\mathcal{K}} \subset \{1, 2, \ldots, J\}$, the average throughput is defined as $\sum_{j \in F_{\mathcal{K}}} m_j / T_{\mathcal{K}}^{\text{total}}$. Apparently, we have

Corollary 2: In a connected wireless network G, if there are J flows originating from $\mathcal{S}_j$, $j = 1, 2, \ldots, J$, each with m_j packets to deliver, and we use the intersession RNC as

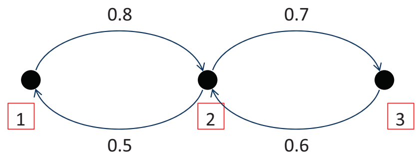

Fig. 3. A three node wireless network.

described here for delivery, the average throughput $\overline{R}_{\mathcal{K}}$ of a set $\mathcal{K}$ subscribing to the same set of flows $F_{\mathcal{K}}$ is given as

$$\overline{R}_{\mathcal{K}} = \frac{\sum_{j\in F_{\mathcal{K}}} m_j}{\sum_{j=1}^{J} m_j} \min_{S\subset \mathrm{Src}(\mathcal{K})} c_{\min}(\cup_{i\in S}\mathcal{S}_i, \mathcal{K}). \qquad (59)$$

In particular, if $F_{\mathcal{K}} = \{1, 2, \ldots, J\}$,

$$\overline{R}_{\mathcal{K}} = \min_{S\subset \mathrm{Src}(\mathcal{K})} c_{\min}(\cup_{i\in S}\mathcal{S}_i, \mathcal{K}). \qquad (60)$$

V. Simulation

Consider a three node wireless network shown in Fig. 3. Assume, based on the underlying MAC, they have the same transmission rate rate of 1 sec^{-1}. The reachability of transmissions is shown by arrows. The labels attached to arrows show the independent reception probabilities. For example, whenever node 2 sends a packet, node 1 and 3 successfully receive it with probability 0.5 and 0.7, respectively. With this information, we may calculate $z_{i,\mathcal{K}}$ for an arbitrary hyperarc $(i, \mathcal{K})$. Assume there are two multicast flows originating from node 1 and node 3, respectively. Node 1 has 200 packets to deliver to node 2 and 3, while node 3 has 300 packets to deliver to node 1 and 2. We use the RNC scheme for multiple flows described in this paper, for which we may write the associated DI's:

$$\dot{V}_{\mathcal{K}} \in \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^{+}(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}), \qquad (61)$$

$$\text{B.C.} \quad V_{\mathcal{K}}(0) = \begin{cases} 300, & 1\in\mathcal{K}, 3\notin\mathcal{K}, \\ 200, & 3\in\mathcal{K}, 1\notin\mathcal{K}, \\ 500, & \{1,3\}\subset\mathcal{K}, \\ 0, & \text{o.w.}. \end{cases} \qquad (62)$$

Fig. 4 shows the analytical solution to (61) as well as the simulation results. The analysis matches the simulations closely. Clearly the rank increase at node 1 should be subject to its min cut bound $c_{\min}(3, 1) = 0.5\ \mathrm{sec}^{-1}$ and node 3 subject to $c_{\min}(1, 3) = 0.7\ \mathrm{sec}^{-1}$. Consequently, $T_1^{\mathrm{total}} = 300/0.5 = 600$ (sec) and $T_3^{\mathrm{total}} = 200/.7 = 285.7$ (sec). For node 2, the flow from node 1 cannot exceed $c_{\min}(1, 2)$; the flow from node 3 cannot exceed $c_{\min}(3, 2)$; and the flow from the ensemble

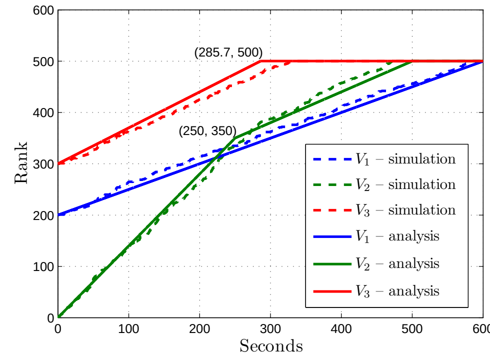

Fig. 4. Two multicast sessions with two sources.

of node 1, 3 cannot exceed $c_{\min}(\{1, 3\}, 2)$. Therefore,

$$T_2^{\mathrm{total}} = \max\{m_1/c_{\min}(1, 2), m_2/c_{\min}(3, 2), (m_1 + m_2)/c_{\min}(\{1, 3\}, 2)\} = 500 \text{ (sec)}.$$

These calculations are readily verified in Fig. 4.

VI. Concluding Remarks

We presented the DEDI framework, based on DE's and/or DI's, for analyzing the throughput of RNC in a wireless network. The throughput of an intersession coding scheme for multiple information flows was then analyzed by solving the associated DI's. We gave a numerical example to demonstrate the accuracy of the DEDI framework and the validity of the theoretical results we obtained. Compared with existing analytical tools, DEDI is much easier to manipulate and applicable to very general settings. We expect that the DEDI framework will be helpful for the advancement of network coding research and network design practice.

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