Signal Space Partitioning Versus Simultaneous Water Filling for Mutually Interfering Systems

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***Abstract*— We consider a communication system with multiple independent user-base pairings in a white Gaussian noise environment, and for which a simultaneous water filling condition is satisfied by users at their respective bases. We focus on the low mutual interference case for which the simultaneous water filling solution is unique and users overlap completely in the signal space. We show that when users at other bases are treated as Gaussian noise, simple separation of users in signal space usually offers much better performance than simultaneous water filling. We then present a distributed algorithm which iteratively moves users toward greater separation in the signal space.**

I. Introduction

In wireless communication systems, nodes are distributed over some region and at any given instant, active nodes are either transmitters or receivers. For simplicity we will assume a one to one mapping between transmitters and receivers, but this condition can be relaxed with no loss of generality. We will call the receivers “bases” and transmitters “users.” Now suppose the spectrum used is unlicensed, then we readily see that any given transmission must cope with interference from other users. That is, when no cooperation among users is assumed, a given user is decoded at its associated base under the interference generated by all the other users. In general, this is an instance of the interference channel [7, p. 382] for which the complete characterization of the capacity region is still an open problem.

An early formulation of the interference channel problem is due to Shannon [19] followed by results obtained decades later by Ahlswede [1], Carleial [3]–[5], Sato [16]–[18], Han and Kobayashi [9], and Costa [6]. While most of these results deal with the strong interference case [3], [4], [9], [17], [18], we note the work of Costa [6] which suggests that weak and moderate interference are more important from a practical perspective. Recent research [21] approaches the Gaussian interference channel from a non-cooperative game theoretic perspective in which users compete for data rates. Each user’s objective is greedy performance maximization without regard for other users in the system, and it is shown [21] that in the case of low interference a simultaneous water filling solution is equivalent to a Nash equilibrium for this Gaussian interference channel game.

More insight into greedy simultaneous water filling distributions that correspond to interference channels with two transmitters and receivers (two user-base pairs) is provided in [13] where a relationship between the geographical distribution of the users and bases (characterized by the user-base gains) and the set of potential water filling solutions is presented. The system, which is depicted in Figure 1, is similar to that considered by Costa [6] and Yu [21] and assumes flat channels for both users to both bases, with the relative gain of each user to its associated base normalized to 1, and to the neighboring base $g_i,\ i = 1, 2$.

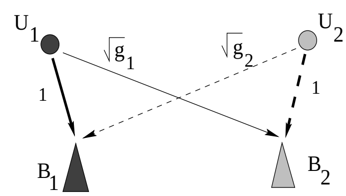
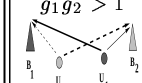
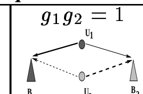
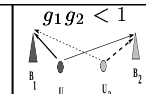


Fig. 1. The system with two transmitters and receivers considered in [13].

Reference [13] shows that three structurally distinct signal space configurations correspond to simultaneous water filling solutions: 1) Complete overlap: users evenly distribute their transmitted energy in all dimensions of the signal space, generating the largest amount of interference; 2) Partial overlap: users share only a subset of the signal space and generate less interference; and 3) No overlap: users reside in orthogonal subspaces and do not interfere with each other. It is also shown in [13] that multiple Nash equilibria are possible. Table 1 summarizes the results in [13] and relates the number of potential Nash equilibria and overlap scenarios to the relative gains of users to bases.

TABLE I

Overlap	**Equilibrium Points**		
	$g_1 g_2 > 1$ 	$g_1 g_2 = 1$ 	$g_1 g_2 < 1$ 
Complete	*unique*	*unique*	*unique*
Incomplete	*many*	*many*	-
None	*many*	*unique*	-

In this paper we focus on “low interference” where $g_1 g_2 < 1$ and where simultaneous water filling represents a *unique* Nash

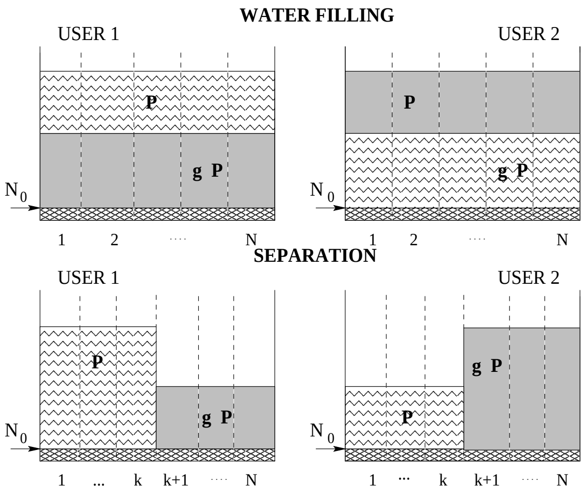


Fig. 2. Simultaneous water filling and signal space partitioning for a symmetric system with two mutually interfering users

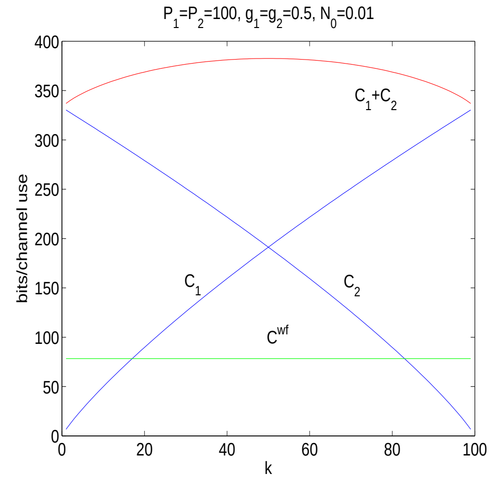


Fig. 3. Capacity variations as a function of user subspace width

equilibrium. We compare this solution to simple and fair allocation of signal space over users, and show that better performance is possible through separation. We also propose a distributed algorithm – or more precisely, an *etiquette* – that moves a symmetric system from complete overlap between users (corresponding to simultaneous water filling) to separation of users in signal space, thus improving each user's rate.

II. Simultaneous Water Filling: An Inefficient Nash Equilibrium

In game theory, a Nash equilibrium is defined by a set of strategies such that each player's strategy is an optimal response to the other players' strategies [8, p. 11]. From this perspective, a Nash equilibrium is reached for the Gaussian interference channel game if and only if a simultaneous water filling solution is satisfied for both users [21], and the optimal strategy of each user is to water fill the signal space regarding the other user as noise. Furthermore, a Nash equilibrium is said to be Pareto deficient (or non-Pareto-optimal) if at least one player would do better and the other one would do no worse by switching to a different strategy [22, p. 52]. Such Nash equilibria are not necessarily *efficient* in that there exist cooperative strategies where both players achieve better returns – a classical example is the *Prisoner's Dilemma* [22, p. 51] and *tit for tat* strategies [10].

Let us consider the symmetric system with two user-base pairs as in Figure 1, with equal user powers P, equal gains g from one user to the neighboring base, and equal background noise level at each base assumed white and Gaussian with variance N_0. Let N be the dimensionality of the signal space. It has been proven in [13] that when a simultaneous water filling distribution is satisfied, both user transmit covariance matrices have the same eigenvectors. Thus, the simultaneous water filling distribution can be graphically illustrated as in the upper diagram of Figure 2, and we will compare it with the signal space partitioning between the two users illustrated in the lower diagram of Figure 2.

With the simultaneous water filling distribution each user achieves the capacity

$$C^{wf} = \frac{N}{2}\log\left(1 + \frac{P}{gP + NN_0}\right) \tag{1}$$

while in the case of signal space partitioning user capacities are given by

$$C_1 = \frac{k}{2}\log\left(1 + \frac{P}{kN_0}\right) \tag{2}$$

$$C_2 = \frac{N-k}{2}\log\left(1 + \frac{P}{(N-k)N_0}\right) \tag{3}$$

with k being the number of dimensions occupied by user 1.

We note that as the number of signal space dimensions k occupied by user 1 increases, C_1 increases and C_2 decreases. For the symmetric system under consideration the optimum point corresponds to equal partitioning of the signal space, for which both user capacities are equal, and the *collective capacity*[1] [13]

$$\mathcal{C} = C_1 + C_2 =$$

$$= \frac{k}{2}\log\left(1 + \frac{P}{kN_0}\right) + \frac{N-k}{2}\log\left(1 + \frac{P}{(N-k)N_0}\right) \tag{4}$$

is maximized. This can be observed in Figure 3 where the capacity that corresponds to the simultaneous water filling solution C^{wf} is compared to C_1 and C_2 for a system with $N = 100$, $P = 100$, $N_0 = 0.01$, $g = 0.5$, and k ranging from 1 to 99 dimensions[2]. We note that for a wide range of values for k both users achieve higher rates C_1 and C_2 if they partition the signal space than C^{wf} corresponding to the simultaneous water filling solution. We also note that the collective capacity $\mathcal{C}$ is maximized when users span orthogonal subspaces of equal dimension, and that $\mathcal{C}$ does not vary significantly over the range of values k for which C_1 and C_2 are larger than C^{wf}.

[1]The term *collective capacity* is used to distinguish it from the information-theoretic *sum capacity* used in related work on multibase systems [12], [15].

[2]The values $k = 1$, respectively $k = 99$, correspond to the extreme cases in which user 1, respectively user 2, reside in only one signal dimension.

We define the point for which the collective capacity $\mathcal{C}$ is maximized as the social optimum for the system.

We can also compare simultaneous water filling and signal space partitioning as a function of the gain g. In the case of simultaneous water filling $\mathcal{C}$ is given by

$$\mathcal{C}_{wf} = N \log\left(\frac{\rho}{\rho g + 1} + 1\right) \quad (5)$$

where $\rho = P/(NN_0)$ represents the *raw SNR* of each user. When users partition the signal space in equal orthogonal subspaces we have

$$\mathcal{C}_{sep} = C_1 + C_2 = \frac{N}{2}\log(2\rho + 1) \quad (6)$$

We note that simultaneous water filling collective capacity $\mathcal{C}_{wf}$ depends on both the gain g and the raw SNR per user ρ, whereas signal space partitioning collective capacity $\mathcal{C}_{sep}$ depends only on ρ.

In illustration, Figure 4 plots $\mathcal{C}_{wf}$ and $\mathcal{C}_{sep}$ as a function of the gain g and user SNR in dB for a signal space with $N = 100$ dimensions, user power $P = 100$, gain $g = 0.1 \ldots 0.9$, and background noise level $N_0 = 10^{-6} \ldots 1$. We note that only for small values of the gain g and user SNR ρ the simultaneous water filling does slightly better than signal space partitioning. Otherwise, the collective capacity under signal space partitioning is larger than it is under simultaneous water filling solution.

To conclude, we note that by comparing the collective capacity values $\mathcal{C}_{sep}$ and $\mathcal{C}_{wf}$ corresponding to the two scenarios for a given gain, signal space partitioning does better than the simultaneous water filling solution, that is $\mathcal{C}_{sep} > \mathcal{C}_{wf}$, when the raw SNR ρ satisfies

$$\rho > \frac{1-2g}{2g^2} \quad (7)$$

Thus, the raw SNR criterion can be used to choose between simultaneous water filling and signal space partitioning.

III. The Social Optimum for a Multiple User/Base System

The analysis presented in the previous section which compares the collective capacity achieved by signal space partitioning with that corresponding to simultaneous water filling can be extended to any number of users M. While a general rigorous analysis of the M user case is complex and requires knowledge of all gains g_{ij} between users and bases implied by the geographic distribution of users/bases, we present for simplicity the analysis for a symmetric system characterized by the same value g for all gains. We note that in the case of "low interference" for multiple users all products $g_{ij}g_{ji} < 1$, and implies a unique simultaneous water filling solution in which all users overlap in all signal space dimension [11]. For the symmetric system case this implies that simultaneous water filling in which users overlap in all signal dimensions is obtained for $g^2 < 1$. For such a system, the expressions of collective capacity in equations (5) and (6) become

$$\mathcal{C}_{wf} = M\frac{N}{2}\log\left[\frac{\rho}{(M-1)\rho g + 1} + 1\right] \quad (8)$$

and

$$\mathcal{C}_{sep} = \frac{N}{2}\log(M\rho + 1) \quad (9)$$

The raw SNR criterion used to compare simultaneous water filling with signal space partitioning in this case implies that $\mathcal{C}_{sep} > \mathcal{C}_{wf}$ when

$$g > \frac{1}{M-1}\left(\frac{1}{\sqrt[M]{1+M\rho} - 1} - \frac{1}{\rho}\right) \quad (10)$$

Using equations (8) and (9) one can obtain plots similar to those in Figure 4 showing again that only for small values of gain and SNR does simultaneous water filling do slightly better than signal space partitioning. Otherwise, the collective capacity under signal space partitioning is larger than for simultaneous water filling, and the difference between them becomes more significant as the number of users M increases. In illustration we present plots for $M = 5$ and $M = 100$ in Figures 5 and 6.

Based on the previous analysis, we concentrate on signal space partitioning as a social optimum for a system with M mutually interfering user-base pairs. which implies an orthogonal signaling scheme. We note that orthogonal signaling based on Frequency Division Multiple Access (FDMA) was analyzed in the context of a Gaussian interference channel scenario with two transmitters and receivers in [2] and a cooperative broadcast scenario was proposed to optimize performance.

Let us denote by P_i the power corresponding to user i, and assume the same (white and Gaussian) background noise level at each base with variance N_0. With user i residing in a subspace of dimension k_i, its capacity is expressed as

$$C_i = \frac{k_i}{2}\log\left(1 + \frac{P_i}{N_0 k_i}\right) \quad (11)$$

and the collective capacity becomes

$$\mathcal{C} = C_1 + C_2 + \ldots + C_M = = \frac{k_1}{2}\log\left(1 + \frac{P_1}{N_0 k_1}\right) + \frac{k_2}{2}\log\left(1 + \frac{P_i}{N_0 k_2}\right) + \ldots + + \frac{N - \sum_{i=1}^{M-1} k_i}{2}\log\left(1 + \frac{P_i}{N_0(N - \sum_{i=1}^{M-1} k_i)}\right) \quad (12)$$

The necessary condition for extremum

$$\frac{\partial \mathcal{C}}{\partial k_i} = 0, \qquad i = 1, \ldots, M-1 \quad (13)$$

implies that the collective capacity is maximized when

$$\frac{P_1}{k_1} = \frac{P_2}{k_2} = \ldots = \frac{P_M}{N - \sum_{i=1}^{M-1} k_i} \quad (14)$$

We note that a similar result was obtained in [2] using a different performance criterion than the collective capacity used in this work and [13]. In light of this result we observe that more energetic users require a larger subspace, or equivalently more bandwidth, at the socially optimal point – a sort of "*Might makes right*" property also seen in the case of optimal signal constellations for single base systems where *oversized*

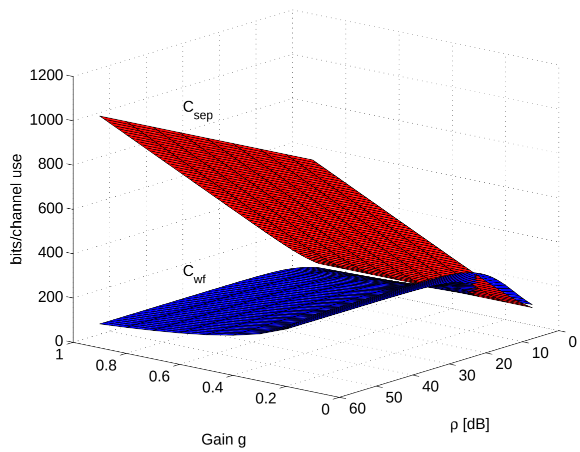


Fig. 4. Water-filling and separation collective capacity as a function of gain and noise level for a symmetric system with $M = 2$ users

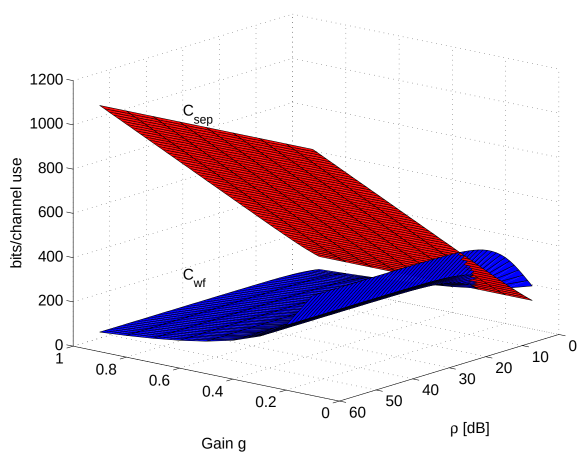


Fig. 5. Water-filling and separation collective capacity as a function of gain and noise level for a symmetric system with $M = 5$ users

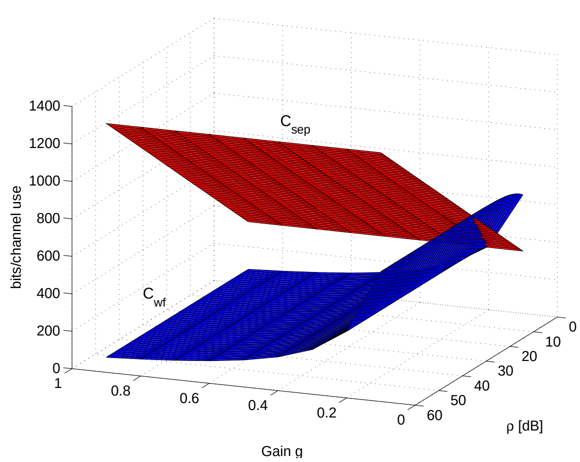


Fig. 6. Water-filling and separation collective capacity as a function of gain and noise level for a symmetric system with $M = 100$ users

users command private channels for communication [14], [20]. We also note that for a symmetric system with equal user powers, the social optimum implies the natural partitioning of the signal space in orthogonal subspaces of equal dimension.

IV. A Distributed Splitting Algorithm

We consider an M-user symmetric system with equal user powers and identical white background noise and note that the greedy optimum has white user spectra [11], [13]. We seek an algorithm which moves a system from simultaneously water filled to spectrally equipartitioned without explicit signaling or coordination between users – though some implicit cooperation is assumed since the equipartitioned solution is Nash-unstable. For this reason, the algorithm is essentially an *etiquette* by which users of the system choose to abide.

We assume an-interference limited systems where the background noise is assumed much smaller than interfering signals. This condition serves double duty in that the raw signal to noise ratio is chosen large enough to satisfy the condition of equation (10). Thus, if some dimension of signal space is devoid of interfering user signal power, then noise in that dimension will be at the *ambient* level and this fact will be observable by all user-base pairs. From a practical perspective, such *occupied* dimensions (or subspaces) can be readily identified using signal covariance methods.

Now consider a two-phase algorithm performed sequentially by each user assuming initially white spectra:

Splitting Algorithm

1) Set $k = 1$
2) Identify a signal space representation (eigenvectors).
3) Vacate $\frac{1}{k+1}$ of the dimensions in which resident, choosing first those dimensions with largest other-user energy
4) Wait for other users to finish
5) Set $k = k + 1$
6) If no interferers in resident space, STOP
7) Else, go to 2.

An illustration of the procedure is provided in FIGURE 7 for $M = 3$ users. We see that users monotonically reduce their signal space occupancy until the stopping criterion is met. Specifically, each user occupies $\frac{1}{k+1}$ of the entire signal space at the completion of step k since

$$\prod_{n=1}^{k}(1-\frac{1}{n+1}) = \prod_{n=1}^{k}\frac{n}{n+1} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \qquad (15)$$

In addition, users actively avoid dimensions with greatest occupancy.

From a more practical perspective, so long as the users update asynchronously and none updates twice in a given cycle, sequencing could be accomplished by a two phase random update procedure where users decided when to vacate portions of the signal space according to (say) an exponential distribution with mean $1/\mu_u$ and then waited another (deterministic) characteristic time $T \gg> 1/\mu_u$ before all subsequent updates until the stopping rule pertained. A formal proof of convergence is provided in [11].

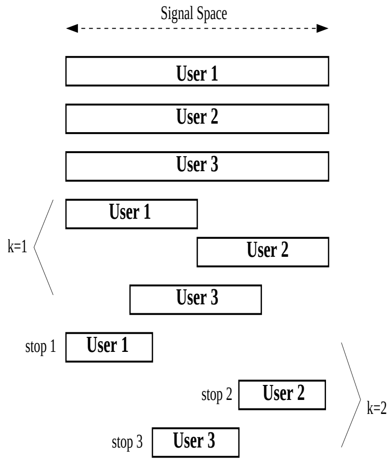


Fig. 7. The splitting algorithm for $M = 3$ users. In step $k = 1$, users 1, 2 and 3 each retreat from half the signal space – which half is arbitrary but shown simply-connected for clarity. In step $k = 2$, users 1 and 2 then retreat from one third of their current occupancy which in this case is only a portion of that which they share with user 3. User 3 then retreats from the dimensions which overlap users 1 and 2 to fully complete the process.

V. Conclusions

In this paper we have focused on the simultaneous water filling solution for a Gaussian interference channel with low interference. We have shown that this solution is often Pareto deficient and that orthogonal signaling offers better performance and leads to a socially optimal point in such cases. To decide Pareto deficiency, we have defined a raw SNR criterion that can be used to choose between simultaneous water filling and signal space partitioning. We have also proposed a distributed algorithm (etiquette) that moves a symmetric system from simultaneously water filled to fair orthogonal signal space partitions.

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