Interference Avoidance and Collaborative Multibase Systems, Part II: flat channels, TSC and algorithmic simplifications

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Abstract

As in the immediately preceeding companion paper, we consider distributed iterative interference avoidance algorithms for uplink wireless systems where base stations share information (collaborate). Here we investigate simplifications possible under a flat channel assumption where the gains are identical for all signal space dimensions over the channel between a given user and a given base. Flat channels lead to structural covariance properties which both simplify calculation of sum capacity, and in addition, reduce the complexity of interference avoidance algorithms. The results also suggest that steering energy selectively at the bases using simple directive antennas might be used as an additional adaptation method to both increase sum capacity and to render uniform shared signal to interference/noise ratios to all users.

Contents

1 Introduction

In this companion paper to [1] we consider a special case where the gains between each user and each base are identical across signal space dimensions. In a frequency division scheme this would correspond to the physical properties of closely spaced channels (within the cohenence bandwidth) where line of sight propagation predominated. Such "flat" channels could arise in a variety of settings where a single path (not necessarily line of sight (LOS)) predominates between transmitters and receivers. We will show how such flat channels lead to structural properties for the received covariance which can be exploited by suitable modifications of the interference avoidance algorithm. We will find that for flat channels, interference avoidance might be used not only for sum capacity capacity maximization, but perhaps more importantly for uniformization of signal to interference/noise ratios (SINRs) for all users under suitable received power constraints.

2 Flat Channel Simplifications

In the previous paper [1] we considered general gain matrices $\mathbf{G}_\ell$. For simplicify, we first consider the simplest of flat channels where the gain between a given user and a given base is identical over all signal space dimensions and derive structural properties of C_{sum}-maximizing $\mathbf{R}$. Later, we will adapt the results to include carrier phase rotation induced by different pathlengths between receivers and transmitters. So, our gain matrices are

$$\mathbf{G}_\ell = \begin{bmatrix} g_{\ell 1}\mathbf{I} \\ \vdots \\ g_{\ell B}\mathbf{I} \end{bmatrix} \tag{1}$$

which results in

$$\mathbf{R}(\ell) = \begin{bmatrix} g_{\ell 1}^2 \mathbf{S}_\ell \mathbf{S}_\ell^\top & \cdots & g_{\ell 1} g_{\ell B} \mathbf{S}_\ell \mathbf{S}_\ell^\top \\ \vdots & \ddots & \vdots \\ g_{\ell B} g_{\ell 1} \mathbf{S}_\ell \mathbf{S}_\ell^\top & \cdots & g_{\ell B}^2 \mathbf{S}_\ell \mathbf{S}_\ell^\top \end{bmatrix} \tag{2}$$

and

$$\mathbf{R} = \sum_\ell \mathbf{R}(\ell) + \mathbf{W} \tag{3}$$

where $\mathbf{W}$ is the noise covariance.[1] We can now state some useful properties of the system covariance matrix $\mathbf{R}$ under a flat channel assumption:

1. $\mathbf{R}$ is composed of $N \times N$ sub-blocks
2. Each sub-block, $\mathbf{R}_{ij}$ satisfies

$$\text{Trace}\left[\mathbf{R}_{ij}\right] = \sum_{\ell} g_{\ell i} g_{\ell j} \text{Trace}\left[\mathbf{S}_\ell \mathbf{S}_\ell^\top\right] + \text{Trace}\left[\mathbf{W}_{ij}\right] = E_{ij} = E_{ji} \tag{4}$$

As usual, we seek to maximize $|\mathbf{R}|$. We recall that a positive definite $N \times N$ matrix $\mathbf{X}$ with constant trace has maximum $|\mathbf{X}|$ when $\mathbf{X} = \frac{\text{Trace}[\mathbf{X}]}{N}\mathbf{I}$ [2]. Unfortunately, unless the off-diagonal blocks of $\mathbf{R}$ have zero trace, the covariance $\mathbf{R}$ *cannot* be a scaled identity matrix. So, we seek the structure of $\mathbf{R}$ which maximizes the determinant subject to the imposed block trace constraints.

First consider a single user and white noise/interference. It is easy to see that such a user sees N identical channels corresponding to the N signal space dimensions. It is intuitively obvious following a water filling argument [3] that such a user should place equal amounts of power in each dimension resulting in a scaled identity codeword covariance matrix $\mathbf{S}\mathbf{S}^\top$ and therefore in $\mathbf{R}$ being composed of $N \times N$ scaled identity sub-blocks. However, with more than one user and colored noise/interference, the optimal structure of $\mathbf{R}$ seems less obvious. Thus, we will derive a general upper bound on the determinant for matrices sharing the structure of $\mathbf{R}$ and sufficient conditions such that the bound is met.

2.1 Bounds on $|\mathbf{R}|$

Consider the optimization

$$\max_{\text{Trace}[\mathbf{R}_{ij} = E_{ij}]} |\mathbf{R}| \tag{5}$$

To proceed, we first define a class $\mathcal{Q}_{NJ}$ of positive definite ($\in \mathcal{M}^+$) matrices $\mathbf{Q}$ where the subscript N denotes the size of the square sub-blocks and J denotes the number of vertical and horizontal sub-blocks $\mathbf{Q}_{ij}$ with trace constraint $\text{Trace}\left[\mathbf{Q}_{ij}\right] = E_{ij}$. Since $\mathbf{Q} \in \mathcal{M}^+$ we also have $\mathbf{Q}_{ij} = \mathbf{Q}_{ji}^\top$. For this class of matrix we have:

[1]Noise in this context could mean Gaussian other-system interference. For a more careful definition of the structure of $\mathbf{W}$, please see the companion paper [1].

Theorem 1 *Let*

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \cdots & \mathbf{Q}_{1J} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{Q}_{J1} & \cdots & \cdots & \mathbf{Q}_{JJ} \end{bmatrix} \tag{6}$$

where $\mathbf{Q} \in \mathcal{Q}_{N,J}$*. Let the trace constraints of the sub-blocks be Trace* $[\mathbf{Q}_{ij}] = E_{ij}$*. Then the determinant of* $\mathbf{Q}$ *is maximized when*

$$\mathbf{Q}_{ij} = \frac{E_{ij}}{N}\mathbf{I}_N \tag{7}$$

so that

$$\max_{\mathbf{Q}\in\mathcal{Q}_{N,J}} |\mathbf{Q}| = \frac{1}{N^{NJ}} |\mathbf{E}|^N \tag{8}$$

where the elements of the symmetric $J \times J$ *matrix* $\mathbf{E}$ *are* $\{E_{ij}\}$*.*

We prove Theorem 1 in the following sections using a recursive approach.

2.2 The Kernel of Recursion

Let $\mathbf{Q} \in \mathcal{Q}_{N,J+1}$ be

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \tag{9}$$

where $\mathbf{A}$ is $NJ \times NJ$, $\mathbf{C}$ is $N \times N$ and $\mathbf{B}$ is $N \times NJ$. We will derive structural properties on any $\mathbf{Q}$ which has maximum $|\mathbf{Q}|$.

We may factor $\mathbf{Q}$ as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{B}^\top \\ 0 & \mathbf{C} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^\top \end{bmatrix} \tag{10}$$

so that

$$|\mathbf{Q}| = |\mathbf{A}||\mathbf{C} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^\top| \tag{11}$$

The term $\frac{\mathbf{Q}}{\mathbf{A}} \equiv \mathbf{C} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^\top$ is called the Schur complement of $\mathbf{A}$ in $\mathbf{Q}$ and if $\mathbf{Q} \in \mathcal{M}^+$ then $\frac{\mathbf{Q}}{\mathbf{A}} \in \mathcal{M}^+$ [4]. We then note that since $|\mathbf{Z}| \leq \left[\frac{\text{Trace}[Z]}{N}\right]^N$ with equality **iff** $\mathbf{Z} \propto \mathbf{I}$ for any $N \times N$

matrix with a trace constraint [2] we must have

$$|\mathbf{Q}| \leq |\mathbf{A}| \left[\frac{\text{Trace}\left[\mathbf{C}\right] - \text{Trace}\left[\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\top}\right]}{N} \right]^{N} \leq |\mathbf{A}| \left[\frac{\text{Trace}\left[\mathbf{C}\right] - \min_{\mathbf{B}} \text{Trace}\left[\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\top}\right]}{N} \right]^{N} \tag{12}$$

with equality **iff** $(\mathbf{C} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\top}) \propto \mathbf{I}$. We therefore turn our attention to minimizing Trace $\left[\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\top}\right]$ in $\mathbf{B}$.

Consider that

$$\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\top} = \mathbf{U}\mathbf{D}\mathbf{V}^{\top}\mathbf{\Psi}\mathbf{\Omega}^{-1}\mathbf{\Psi}^{\top}\mathbf{V}\mathbf{D}^{\top}\mathbf{U}^{\top} \tag{13}$$

where $\mathbf{A} = \mathbf{\Psi}\mathbf{\Omega}\mathbf{\Psi}^{\top}$, an eigendecomposition and $\mathbf{B} = \mathbf{U}\mathbf{D}\mathbf{V}^{\top}$ the usual singular value decomposition with

$$\mathbf{D} = \left[\begin{array}{cc} \mathcal{D} & \mathbf{0} \end{array} \right] \tag{14}$$

where $\mathcal{D}$ is an $N \times N$ diagonal matrix containing the singular values of $\mathbf{B}$. We then have

$$\text{Trace}\left[\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\top}\right] = \text{Trace}\left[\mathbf{D}\mathbf{V}^{\top}\mathbf{\Psi}\mathbf{\Omega}^{-1}\mathbf{\Psi}^{\top}\mathbf{V}\mathbf{D}^{\top}\right] = \text{Trace}\left[\mathbf{D}^{\top}\mathbf{D}\mathbf{P}^{\top}\mathbf{\Omega}^{-1}\mathbf{P}\right] \tag{15}$$

after defining $\mathbf{V}^{\top}\mathbf{\Psi} = \mathbf{P}^{\top}$. This expression reduces to

$$\text{Trace}\left[\mathbf{D}^{\top}\mathbf{D}\mathbf{P}^{\top}\mathbf{\Omega}^{-1}\mathbf{P}\right] = \sum_{ij} p_{ij}^2 \frac{d_j^2}{\omega_i} \tag{16}$$

where the d_j^2 are the elements of the diagonal matrix $\mathbf{D}^{\top}\mathbf{D}$ of which N are the squared singular values of $\mathbf{B}$ and the rest are zero. The p_{ij} are the elements of the $\mathbf{P}$ matrix. Since $\mathbf{P}$ is the product of two unitary matrices, it is itself a unitary matrix so that

$$\sum_i p_{ij}^2 = \sum_j p_{ij}^2 = 1 \tag{17}$$

So, the problem is now

$$\min_{\mathbf{B}} \text{Trace}\left[\mathbf{D}^{\top}\mathbf{D}\mathbf{P}^{\top}\mathbf{\Omega}^{-1}\mathbf{P}\right] = \min_{\{p_{ij}, d_i\}} \sum_{ij} p_{ij}^2 \frac{d_j^2}{\omega_i} \tag{18}$$

Now, consider that via equation (17) p_{ij}^2 is a probability mass function in i and also in j. Therefore, we can rewrite the optimization as

$$\min_{\{p_{ij}, d_i\}} \sum_i \frac{E_i[d^2]}{\omega_i} \tag{19}$$

where

$$E_i[d^2] = \sum_j p_{ij}^2 d_j^2 \tag{20}$$

Assume with no loss of generality that the d_j^2 and the ω_i are fixed and ordered from largest to smallest. We first note that

$$\sum_i E_i[d^2] = \sum_j d_j^2 \tag{21}$$

and

$$d_{JN}^2 \leq E_i[d^2] \leq d_1^2 \tag{22}$$

Now for the moment, imagine we can choose any $E_i[d^2]$ we'd like subject to the constraints of equation (21) and equation (22). That is, ignore the underlying structure of the $E_i[d^2]$ as generated from squared entries of a unitary matrix. We then note that if $E_1[] \neq d_1^2$, we can always reduce Trace $\left[\mathbf{D}^\top \mathbf{D} \mathbf{P}^\top \mathbf{\Omega}^{-1} \mathbf{P}\right]$ by reducing one of the other $E_i[d^2]$ (which multiplies another $1/\omega_i > 1/\omega_1$) by some ϵ and adding ϵ to $E_1[d^2]$. If no such greater $1/\omega_i$ exists, then we can still increase $E_1[d^2]$ to its maximum with no penalty to the objective. For completeness, please note that if $E_1[d^2] < d_1^2$ then the sum constraint on the $E_i[d^2]$ in equation (21) guarantees there exists at least one $E_i[d^2]$ from which ϵ can be "borrowed."

Thus, at any minimum we must have $E_1[d^2] = d_1^2$. However, to do so implies that $p_{11} = 1$ which implies that $p_{1j} = 0$, $j \neq 1$ and that $p_{i1} = 0$, $i \neq 1$ – further implying that $E_i[d^2] \leq d_2^2$, $i > 1$. Proceeding recursively, we see that to minimize Trace $\left[\mathbf{D}^\top \mathbf{D} \mathbf{P}^\top \mathbf{\Omega}^{-1} \mathbf{P}\right]$, we will have

$$p_{ij} = \begin{cases} 1 & i = j, j = 1, 2, \cdots, JN \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

so that

$$\min_{\mathbf{B}} \text{Trace}\left[\mathbf{D}^\top \mathbf{D} \mathbf{P}^\top \mathbf{\Omega}^{-1} \mathbf{P}\right] = \min_{\{d_j\}} \sum_{i=1}^{N} \frac{d_i^2}{\omega_i} \tag{24}$$

Equation (23) implies that the right singular basis set of $\mathbf{B}$ and the eigenvector matrix of $\mathbf{A}$ should be **identical**[2] and that the singular values d_i of $\mathbf{B}$ should be ordered from largest to smallest magnitude if the eigenvalues of $\mathbf{A}$ are ordered from largest to smallest.

[2] $\mathbf{V} = \mathbf{\Psi}$ in equation (13).

Now, consider the optimization over the $\{d_j\}$ of equation (24). Owing to the trace constraint on sub-blocks of $\mathbf{Q}$ we have

$$\mathbf{B} = \left[\begin{array}{cc} \mathbf{B}_1 & \cdots \mathbf{B}_J \end{array} \right] \tag{25}$$

with each $\mathbf{B}_j$ an $N \times N$ matrix and with Trace $[\mathbf{B}_j] = E_{\mathbf{B}_j}$. We then note that

$$\mathbf{B}\mathbf{B}^\top = \mathbf{U}\mathcal{D}^2\mathbf{U}^\top \tag{26}$$

However, owing to the structure of $\mathbf{B}$ we also have

$$\mathbf{B}\mathbf{B}^\top = \sum_j \mathbf{B}_j\mathbf{B}_j^\top = \sum_j \mathbf{U}_j\mathcal{D}_j\mathbf{U}_j^\top \tag{27}$$

so that

$$\mathcal{D}^2 = \sum_j \mathbf{U}^\top\mathbf{U}_j\mathcal{D}_j\mathbf{U}_j^\top\mathbf{U} = \sum_j \mathbf{Z}_j^\top\mathcal{D}_j\mathbf{Z}_j \tag{28}$$

where $\mathbf{Z}_j = \mathbf{U}^\top\mathbf{U}_j$. This leads to

$$d_i^2 = \sum_{j=1}^{J}\sum_{n=1}^{N} z_{in}^2 d_{jn}^2 \tag{29}$$

where z_{in} are the elements of $\mathbf{Z}$ and thence to

$$\text{Trace}\left[\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^\top\right] = \sum_{i=1}^{N}\frac{d_i^2}{\omega_i} = \sum_{ijn}\frac{z_{in}^2 d_{jn}^2}{\omega_i} \tag{30}$$

Now, consider that

$$\text{Trace}\left[\mathbf{B}_j\right] = \text{Trace}\left[\mathbf{U}_j\mathbf{D}_j\mathbf{V}_j^\top\right] = \text{Trace}\left[\mathbf{D}_j\mathbf{V}_j^\top\mathbf{U}_j\right] = \text{Trace}\left[\mathbf{D}_j\mathbf{P}_j\right] = E_{\mathbf{B}_j} \tag{31}$$

where as before $\mathbf{P}_j$ is a unitary matrix obtained from the multiplication of two unitary matrices $\mathbf{U}_j$ and $\mathbf{V}_j^\top$ whose elements $p_{in}(j)$ obey equation (17) and for which no entry $p_{in}(j)$ can have magnitude larger than 1. We then have

$$\text{Trace}\left[\mathbf{D}_j\mathbf{P}_j\right] = \sum_n d_{jn}p_{nn}(j) = E_{\mathbf{B}_j} \tag{32}$$

We can now form the constrained optimization of equation (30) as

$$\min_{\{d_{jn}\}}\sum_{ijn}\frac{z_{in}^2(j)d_{jn}^2}{\omega_i} + \sum_j \lambda_j\left(\sum_n p_{nn}(j)d_{jn} - E_{\mathbf{B}_j}\right) \tag{33}$$

with λ_j the constraint constants so that at the extremal points we must have

$$\sum_i \frac{z_{in}^2(j) d_{jn}}{\omega_i} = -\frac{\lambda_j}{2} p_{nn}(j) \tag{34}$$

or

$$d_{jn} = -\frac{\lambda_j}{2} p_{nn}(j) \Big/ \sum_i \frac{z_{in}^2(j)}{\omega_i} \tag{35}$$

Multiplying both sides by $p_{nn}(j)$ and summing over n we have

$$\sum_n d_{jn} p_{nn}(j) = E_{\mathbf{B}_j} = -\frac{\lambda_j}{2} \sum_n p_{nn}^2(j) \Big/ \sum_i \frac{z_{in}^2(j)}{\omega_i} \tag{36}$$

so that

$$\lambda_j = -2E_{\mathbf{B}_j} \Big/ \sum_n p_{nn}^2(j) \frac{1}{\sum_i \frac{z_{in}^2(j)}{\omega_i}} \tag{37}$$

We then re-evaluate the metric in equation (30) and obtain

$$\sum_{jn} d_{jn}^2 \sum_i \frac{z_{in}^2(j)}{\omega_i} = \sum_j \frac{\sum_n E_{\mathbf{B}_j} d_{jn} p_{nn}(j)}{\sum_n p_{nn}^2(j) \frac{1}{\sum_i \frac{z_{in}^2(j)}{\omega_i}}} = \sum_j \frac{E_{\mathbf{B}_j}^2}{\sum_n p_{nn}^2(j) \frac{1}{\sum_i \frac{z_{in}^2(j)}{\omega_i}}} \tag{38}$$

We first notice that minimization of equation (38) can be cast as separate minimizations for each j. So, we seek to maximize

$$\max_{\{p_{nn}(j), z_{in}(j)\}} \sum_n p_{nn}^2(j) \frac{1}{\sum_i \frac{z_{in}^2(j)}{\omega_i}} \tag{39}$$

We then note that equation (39) is increasing in the $p_{nn}^2(j)$, so at the maximum we must have all $p_{nn}^2(j) = 1$. Thus, we must have $\mathbf{U}_j = \mathbf{V}_j$ – i.e., each $\mathbf{B}_j$ should be symmetric with eigenvalues d_{jn}.

Now we notice that

$$\frac{1}{\omega_1} \leq \underbrace{\sum_i \frac{z_{in}^2(j)}{\omega_i}}_{q_n(j)} \leq \frac{1}{\omega_N} \tag{40}$$

and also that

$$\sum_n q_n(j) = \sum_i \frac{1}{\omega_i} \tag{41}$$

by equation (17). We therefore seek to maximize

$$Q(j) = \sum_n \frac{1}{q_n(j)} \tag{42}$$

Ignoring the detailed structure of the $q_n(j)$ for now and looking only to the constraints of equation (40) and equation (41), we note that $Q(j)$ can always be increased if there exists $q_n(j) > 1/\omega_1$ and another $q_k(j)$ with $q_n(j) < q_k(j) < 1/\omega_N$ by reducing $q_n(j)$ and increasing $q_k(j)$ by ϵ to maintain the sum constraint of equation (41). That is, suppose $x < y$. Then

$$\left(\frac{1}{x-\epsilon} + \frac{1}{y+\epsilon}\right) - \left(\frac{1}{x} + \frac{1}{y}\right) = \frac{\epsilon}{x(x-\epsilon)} - \frac{\epsilon}{y(y+\epsilon)} > 0 \tag{43}$$

since $x(x-\epsilon) < y(y+\epsilon)$.

So, with no loss of generality we set $z_{1n}^2(j) = \delta_{1n}$ which owing to the structure of the unitary matrix $\mathbf{Z}_j$ implies $z_{1k} = 0$, $k > 1$ and $z_{k1} = 0$, $k > 1$. Thus, for the remaining choices of $z_{in}(j)$ we must have

$$\frac{1}{\omega_2} \leq \underbrace{\sum_{i \neq 1} \frac{z_{in}^2(j)}{\omega_i}}_{q_n(j)} \leq \frac{1}{\omega_N} \tag{44}$$

and

$$\sum_{n \neq 1} q_n(j) = \sum_{i \neq 1} 1/\omega_i \tag{45}$$

Recursive application of this procedure leads to

$$z_{in}^2(j) = \delta_{in} \tag{46}$$

and thence to

$$\sum_i \frac{z_{in}^2(j)}{\omega_i} = \frac{1}{\omega_n} \tag{47}$$

and

$$\lambda_j = -2\frac{E_{\mathbf{B}_j}}{\sum\limits_{i=1}^{N} \omega_i} \tag{48}$$

With $p_{nn}^2(j) = 1$ and equation (35) we then have

$$d_{jn} = \omega_n \frac{E_{\mathbf{B}_j}}{\sum\limits_{i=1}^{N} \omega_i} \tag{49}$$

$n = 1, 2, \cdots, N$.

We summarize the results as a theorem:

Theorem 2 *Let* $\mathbf{A}$ *be a* $JN \times JN$ *positive definite matrix with diagonal factorization* $\mathbf{\Phi\Omega\Phi}^\top$ *and* $\mathbf{B}$ *be a matrix composed of* J $N \times N$ *sub-blocks arranged as*

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_J \end{bmatrix} \tag{50}$$

and having Trace $[\mathbf{B}_j] = E_{\mathbf{B}_j}$ *and singular value decomposition* $\mathbf{B} = \mathbf{UDV}^\top$*. Further, assume that the eigenvalues of* $\mathbf{A}$ *and the singular values of* $\mathbf{B}$ *are arranged in descending order.*

Then, Trace $\left[\mathbf{BA}^{-1}\mathbf{B}^\top\right]$ *is minimized* **iff***:*

- $\mathbf{B} = \mathbf{UD\Phi}^\top$*. That is, the right basis singular basis set of* $\mathbf{B}$ *and the eigenvector matrix* $\mathbf{\Phi}$ *are identical.*

- *Each* $\mathbf{B}_j = \mathbf{U}_j\mathbf{D}_j\mathbf{V}_j^\top$ *is symmetric (*$\mathbf{V}_j = \mathbf{U}_j$*) so that the singular values* d_{jn} *are also the eigenvalues.*

- $\mathbf{U}_j = \mathbf{U}$

- *The eigenvalues of* $\mathbf{B}_j$ *satisfy*

$$d_{jn} = \frac{\omega_n}{\sum_{i=1}^{N} \omega_i} E_{\mathbf{B}_j}$$

  where $\omega_1, \cdots, \omega_N$ *are the* N *largest eigenvalues of* $\mathbf{A}$*.*

The resulting minimum value is

$$\min_{\mathbf{B}} \textit{Trace}\left[\mathbf{BA}^{-1}\mathbf{B}^\top\right] = \frac{\sum_j E_{\mathbf{B}_j}^2}{\sum_{i=1}^{N} \omega_i} \tag{51}$$

2.3 Show for $J = 2$

We now return to the factorization of equation (12) and note that since

$$|\mathbf{Q}| \leq \max|\mathbf{A}| \left[\frac{\text{Trace}\left[\mathbf{C}\right] - \min_{\mathbf{B}}\text{Trace}\left[\mathbf{BA}^{-1}\mathbf{B}^\top\right]}{N}\right]^N \tag{52}$$

and $\mathbf{A} \in \mathcal{Q}_{N,J-1}$, further factorizations of each leftmost determinant are possible until we have $N \times N$ $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ at which point the following theorem follows as a consequence of Theorem 2 and equation (11).

Theorem 3 *Let*

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \tag{53}$$

where $\mathbf{Q} \in \mathcal{Q}_{N,2}$ with submatrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ all $N \times N$.

Then $\mathbf{Q} \in \mathcal{Q}_{N,2}$ has maximum determinant **iff** *each $N \times N$ submatrix is a scaled identity matrix.*

Proof: Via Theorem 2, Trace $\left[\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^\top\right]$ is minimized when $\mathbf{B}$ is a scaled replica of $\mathbf{A}$. $|\mathbf{A}|$ is maximized when $\mathbf{A} = \frac{\text{Trace}[\mathbf{A}]}{N}\mathbf{I}$, a scaled identity matrix [2,5]. Via Theorem 2 we then have $\mathbf{B} = \frac{\text{Trace}[\mathbf{B}]}{N}\mathbf{I}$. Likewise, from equation (11) and we see that $\left|\mathbf{C} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^\top\right|$ meets the bound of equation (12) when $\mathbf{C} = \frac{\text{Trace}[\mathbf{C}]}{N}\mathbf{I}$ thus completing the proof. •

2.4 Assume for J and Prove for $J+1$

Suppose following equation (12) that $\mathbf{Q} \in \mathcal{Q}_{N,J+1}$ and that the sub-block traces of $\mathbf{A} \in \mathcal{Q}_{N,J}$ are $\{E_{ij}\}$. Then assume that $|\mathbf{A}|$ is maximized when $\mathbf{A}$ has scaled identity sub-blocks. Scaled identity sub-blocks for $\mathbf{A}$ implies J distinct eigenvalues for $\mathbf{A}$, each repeated N times since $\mathbf{A} = \mathbf{E} \otimes \mathbf{I}_N$, the Kronecker product of the $J \times J$ symmetric matrix $\mathbf{E} = \{E_{ij}\}$ and an $N \times N$ identity matrix [4]. Since the eigenvalues of each $\mathbf{B}_j$ must be scaled replicas of the largest N eigenvalues of $\mathbf{A}$, each $\mathbf{B}_j$ must be a scaled identity matrix. Then, to maximize the determinant in equation (11) we must have $\mathbf{C}$ a scaled identity matrix as well. Thus, the $\mathbf{Q}$ which maximizes $|\mathbf{Q}|$ has each $\mathbf{Q}_{ij} = (\text{Trace}\left[\mathbf{Q}_{ij}\right]/N)\mathbf{I}$. **This completes the proof of Theorem 1.**

Thus, the covariance matrix $\mathbf{R}$ for a system with B bases belongs to the class $\mathcal{Q}_{N,B}$, and to maximize $|\mathbf{R}|$ each of the sub-blocks of $\mathbf{R}$ should be a scaled identity matrix if possible. We now apply Theorem 1 to special cases.

2.5 Sum Capacity Bounds in White Noise/Interference

If we assume independent white noise/interference then each block of $\mathbf{W}$ is of the form $\mathbf{W}_{ij} = \frac{\omega_{ij}}{N}\mathbf{I}_N$, $i, j = 1, \ldots, B$ where $\omega_{ij} = \text{Trace}\,[\mathbf{W}_{ij}]$. From equation (4) we have the sub-block traces of $\mathbf{R}$ as

$$E_{ij} = \sum_{\ell=1}^{L} M_\ell g_{\ell i} g_{\ell j} + \omega_{ij} \tag{54}$$

Direct application of Theorem 1 to $\mathbf{R}$ results in:

$$\sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \mathbf{S}_\ell \mathbf{S}_\ell^\top = \frac{E_{ij} - \omega_{ij}}{N}\mathbf{I}_N \tag{55}$$

which can always satisfied if each of the $\mathbf{S}_k\mathbf{S}_k^\top$ is a scaled identity matrix (although there may be other solutions as well). We state this result as a theorem:

Theorem 4 *For white noise/interference with sub-block traces ω_{ij}, sum capacity is maximized when all the codeword covariances are*

$$\mathbf{S}_k\mathbf{S}_k^\top = \mathbf{X}_k = \frac{M_k}{N}\mathbf{I}_N$$

and the corresponding sum capacity value is

$$C_{\max} = \frac{N}{2}\left[\log|\mathbf{E}| - \log|\boldsymbol{\omega}|\right] \tag{56}$$

where $\boldsymbol{\omega}$ is a $B \times B$ matrix with elements $\{\omega_{ij} = \text{Trace}\,[\mathbf{W}_{ij}]\}$.

2.6 Sum Capacity Bounds in Colored Noise

For colored noise, the trace constraints are identical to equation (55). Theorem 1 requires that

$$\sum_{k=1}^{L} g_{ik} g_{jk} \mathbf{X}_k + \mathbf{W}_{ij} = \frac{E_{ij}}{N}\mathbf{I}_N \tag{57}$$

where $\mathbf{X}_k = \mathbf{S}_k\mathbf{S}_k^\top$. Owing to the structure of the covariance matrix, equation (57) can be interpreted as a set of $\frac{1}{2}B(B+1)$ equations in L unknown covariances $\{\mathbf{X}_k\}$.

The question is whether there exists a realizable/feasible set $\{\mathbf{X}_k\}$ which satisfies equation (57), and if no such set $\{\mathbf{X}_k\}$ exists, what the actual optimizing set should be. We do not consider the latter question here and simply assume that whether a feasible solution to equation (57) exists or not, Theorem 1 provides an upper bound which we state as:

Theorem 5 *For a multiple base system with B basestations, L transmitting locations and $NB \times NB$ noise covariance matrix $\mathbf{W}$, the maximum sum capacity is achieved* **iff** *the codeword ensemble covariances $\mathbf{X}_k = \mathbf{S}_k \mathbf{S}_k^\top$ satisfy*

$$\sum_{k=1}^{L} g_{ik} g_{jk} \mathbf{X}_k + \mathbf{W}_{ij} = \frac{E_{ij}}{N} \mathbf{I}_N \qquad\qquad i, j = 1, 2, \ldots, B$$

and the associated sum capacity value is

$$C_{\max} = \frac{1}{2} \left[N \log |\mathbf{E}| - NB \log N - \log |\mathbf{W}| \right] \tag{58}$$

If no feasible $\{\mathbf{X}_k\}$ exists, then $C_{\max}$ serves as an upper bound.

3 TSC and Interference Avoidance Simplifications

For single receiver systems in white noise, the Total Squared Correlation (TSC) is usually defined as the sum of squared correlations of codewords (signature sequences) at the receiver. For colored noise, it has also been more generally defined as the trace of the squared covariance matrix, Trace $[\mathbf{R}^2]$ [5] and we will use that definition here. We will show that under certain circumstances, minimizing TSC in flat channel multibase systems is equivalent to maximizing sum capacity and that this equivalence allows simplification of the interference avoidance algorithm.

3.1 Bounds on TSC for Multiple Receiver Systems

First we note that

$$\text{Trace}\left[\mathbf{R}^2\right] = \sum_{ij} \text{Trace}\left[\mathbf{R}_{ij} \mathbf{R}_{ji}\right] = \sum_{ij} \text{Trace}\left[\mathbf{R}_{ij}^\top \mathbf{R}_{ij}\right] \tag{59}$$

and then via Theorem 2 with $\mathbf{A} = \mathbf{I}$ we must have

$$\sum_{ij} \text{Trace}\left[\mathbf{R}_{ij}^\top \mathbf{R}_{ij}\right] \geq \frac{1}{N} \left(\text{Trace}\left[\mathbf{R}_{ij}\right]\right)^2 \tag{60}$$

with equality **iff** $\mathbf{R}_{ij} = \frac{\text{Trace}[\mathbf{R}_{ij}]}{N} \mathbf{I}$. So, Trace $[\mathbf{R}^2]$ is minimized **iff** each of its sub-blocks is a scaled identity matrix. We state this result as a theorem:

Theorem 6 *Let $\mathbf{R} \in \mathcal{Q}_{NB}$ with sub-block traces Trace $[\mathbf{R}_{ij}] = E_{ij}$. Then*

$$\textit{Trace}\left[\mathbf{R}^2\right] \geq \sum_{ij} \frac{E_{ij}^2}{N} \tag{61}$$

with equality **iff** *the sub-blocks of $\mathbf{R}$ satisfy $\mathbf{R}_{ij} = \frac{E_{ij}}{N}\mathbf{I}$*

We now draw a parallel between sum capacity maximization and TSC minimization. Notice that if λ_i are the eigenvalues of $\mathbf{R}$, for TSC we seek to minimize

$$\text{Trace}\left[\mathbf{R}^2\right] = \sum_i \lambda_i^2 \tag{62}$$

and for sum capacity we seek to maximize

$$\log|\mathbf{R}| = \sum_i \log \lambda_i \tag{63}$$

Optimization of either metric over $\mathbf{R} \in \mathcal{Q}_{NB}$, a convex class, will result in the same set of optimizing $\mathbf{R}$ via Schur convexity [6, 7]. However, we are not necessarily optimizing over the class of $\mathbf{R} \in \mathcal{Q}_{NB}$, but rather, over the codeword sets $\mathbf{S}_i$ which imposes another set of constraints on $\mathbf{R}$. Therefore, the optimization of TSC and sum capacity will in general be different as illustrated by a simple example.

Consider

$$\mathbf{R} = \mathbf{G}\mathbf{S}\mathbf{S}^\top\mathbf{G}^\top + \mathbf{W} \tag{64}$$

and note that

$$\mathbf{G} = \mathbf{g} \otimes \mathbf{I} = \begin{bmatrix} g_1\mathbf{I} \\ \vdots \\ g_B\mathbf{I} \end{bmatrix} \tag{65}$$

where $\mathbf{g}$ is a B-dimensional vector, $\mathbf{I}$ is an $N \times N$ identity matrix and $\otimes$ is the Kronecker product [4]. We can then define

$$\mathbf{U} = \frac{1}{|\mathbf{g}|}\mathbf{G} \tag{66}$$

and see that $\mathbf{U}$ has orthonormal columns so that $\mathbf{U}^\top\mathbf{U} = \mathbf{I}$. We then define the orthonormal complement of $\mathbf{U}$ as $\bar{\mathbf{U}}$ and note that

$$\begin{bmatrix} \mathbf{U} & \bar{\mathbf{U}} \end{bmatrix} \begin{bmatrix} \mathbf{U}^\top \\ \bar{\mathbf{U}}^\top \end{bmatrix} = \mathbf{I} \tag{67}$$

We can now apply a similarity transform to $\mathbf{R}$

$$\begin{bmatrix} \mathbf{U}^\top \\ \bar{\mathbf{U}}^\top \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{U} & \bar{\mathbf{U}} \end{bmatrix} = \mathbf{R}' = \begin{bmatrix} |\mathbf{g}|^2\mathbf{S}\mathbf{S}^\top + \mathbf{W}'_{11} & \mathbf{W}'_{12} \\ \mathbf{W}'_{21} & \mathbf{W}'_{22} \end{bmatrix} \tag{68}$$

where

$$\mathbf{W}'_{11} = \mathbf{U}^\top\mathbf{W}\mathbf{U} \tag{69}$$

$$\mathbf{W}'_{12} = \mathbf{U}^\top\mathbf{W}\bar{\mathbf{U}} \tag{70}$$

$$\mathbf{W}'_{21} = \bar{\mathbf{U}}^\top\mathbf{W}\mathbf{U} \tag{71}$$

and

$$\mathbf{W}'_{22} = \bar{\mathbf{U}}^\top\mathbf{W}\bar{\mathbf{U}} \tag{72}$$

Under a similarity transform, the eigenvalues of $\mathbf{R}$ are unchanged and we still have $|\mathbf{R}| = |\mathbf{R}'|$ and Trace $[\mathbf{R}^2] =$ Trace $[(\mathbf{R}')^2]$. But via Schur factorization [8] we have

$$\max_{\mathbf{S}} |\mathbf{R}'| = \max_{\mathbf{S}} |\mathbf{W}'_{22}| \left| |\mathbf{g}|^2\mathbf{S}\mathbf{S}^\top + \mathbf{W}'_{11} - \mathbf{W}'_{12}(\mathbf{W}'_{22})^{-1}\mathbf{W}'_{21} \right| \tag{73}$$

whereas

$$\begin{aligned} \min_{\mathbf{S}} \text{Trace}\left[(\mathbf{R}')^2\right] \;&=\; \min_{\mathbf{S}} \text{Trace}\left[(|\mathbf{g}|^2\mathbf{S}\mathbf{S}^\top + \mathbf{W}'_{11})^2\right] \\ &\quad + 2\text{Trace}\left[\mathbf{W}'_{12}\mathbf{W}'_{21}\right] + \text{Trace}\left[(\mathbf{W}'_{22})^2\right] \end{aligned} \tag{74}$$

In both cases optimization requires $\mathbf{S}\mathbf{S}^\top$ to waterfill the covariance with which it appears [1, 5, 9]. But if $\mathbf{W}'_{12}(\mathbf{W}'_{22})^{-1}\mathbf{W}'_{21} \neq 0$, the optimizing $\mathbf{S}\mathbf{S}^\top$ might be different for TSC and sum capacity. Therefore, TSC minimization and sum capacity maximization are in general different problems for multiple base systems.

Nonetheless, there are a variety of different conditions under which the optimization of TSC and sum capacity will be equivalent such as when

$$\mathbf{W}'_{12}(\mathbf{W}'_{22})^{-1}\mathbf{W}'_{21} \propto \mathbf{I} \tag{75}$$

or when the noise covariance has scaled identity sub-blocks among others. However, we will opt for the simplest operational requirement – so long as $\mathbf{R}$ can be realized with scaled identity sub-blocks, maximizing sum capacity will be equivalent to minimizing TSC, owing to Theorem 1 and Theorem 6. Heuristically, we can always be assured that such realizations are possible when the noise covariance $\mathbf{W}$ has white (scaled identity) sub-blocks, as well as being reasonably assured when the signal energies are sufficiently large in each sub-block that complete waterfilling over the fixed colored noise is possible.

We refine these general notions by noting that if Trace $[\mathbf{R}^2]$ meets the bound of Theorem 6 we require

$$\sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \mathbf{S}_\ell \mathbf{S}_\ell^\top + \mathbf{W}_{ij} = \frac{E_{ij}}{N}\mathbf{I} \tag{76}$$

This constitutes a set of $\frac{B(B-1)}{2}$ in L unknown covariances. Letting $\mathbf{X}_\ell = \mathbf{S}_\ell \mathbf{S}_\ell^\top$ we then have the following theorem:

Theorem 7 *Minimization of TSC and maximization of sum capacity are equivalent problems if there exists a solution to the* $\frac{B(B-1)}{2}$ *set of equations in* L *unknown covariances* $\mathbf{X}_\ell$

$$\sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \mathbf{X}_\ell + \mathbf{W}_{ij} = \frac{E_{ij}}{N}\mathbf{I} \tag{77}$$

such that each $\mathbf{X}_\ell$ *is positive semidefinite and Trace* $[\mathbf{X}_\ell] = M_\ell$*.*[3]

In all that follows we will assume that the conditions of Theorem 7 can be satisfied.

3.2 Subspace Interference Avoidance

Equation (68) is the basis for a simple interference avoidance algorithm. Specifically,

The Flat Channel Eigen-Algorithm

1. Start with a random codeword ensemble $\{\mathbf{S}_k\}$ and a specified set of transmit locations which determine the gain vectors $\{\mathbf{g}_k\}$

2. For some user k, compute $\mathbf{U}_k$ as in equation (66)

3. Compute

$$\mathbf{A}_k = \mathbf{U}_k^\top \left(\mathbf{R} - \mathbf{G}_k \mathbf{S}_k \mathbf{S}_k^\top \mathbf{G}_k^\top\right) \mathbf{U}_k \tag{78}$$

[3] We have assumed that $\mathbf{S}$ has M_ℓ unit norm columns.

4. Replace codeword i of user k ($\mathbf{s}_{ki}$) by a minimum eigenvector of

$$\mathbf{S}_k\mathbf{S}_k^\top - \mathbf{s}_{ki}\mathbf{s}_{ki}^\top + \frac{1}{|\mathbf{g}_k|^2}\mathbf{A}_k \tag{79}$$

5. Repeat in some reasonable sequence[4] of k and i. Use escape procedures if necessary [9].

6. Stop when within some tolerance of the optimal fixed point.

Since at each step interference avoidance will decrease TSC, and since TSC is bounded from below, the algorithm must converge in TSC. In addition, a proof similar to those found in [1,9] can be formulated to show that all codewords become eigenvectors of their respective $\mathbf{S}_k\mathbf{S}_k^\top - \mathbf{s}_{ki}\mathbf{s}_{ki}^\top + \frac{1}{|\mathbf{g}_k|^2}\mathbf{A}_k$ under a specific update sequence called **Greedy+** interference avoidance [1, 9]. This in turn implies that each $\mathbf{S}_k\mathbf{S}_k^\top$ waterfills its respective $\mathbf{A}_k/|\mathbf{g}_k|^2$. That is, at the equilibrium we will have

$$\left[\mathbf{S}_k\mathbf{S}_k^\top + \mathbf{A}_k/|\mathbf{g}_k|^2\right]\mathbf{S}_k = \alpha_k\mathbf{S}_k \tag{80}$$

for α_k some constant. If not, then escape procedures [9] can be applied until equation (80) is true $\forall k$.

Thus, the stopping rule consists of evaluating when all covariances simultaneously waterfill their respective $\mathbf{A}_k/|\mathbf{g}_k|^2$ to within some tolerance, and it is obvious that such simultaneous waterfilling is a necessary condition for the optimum codeword ensemble lest we have a contradiction. Furthermore, Trace $[\mathbf{R}^2]$ is strictly convex. That is, for $0 \le \lambda \le 1$ and $\mathbf{R}_1 \ne \mathbf{R}_2$ we can show

$$\text{Trace}\left[(\lambda\mathbf{R}_1 + (1-\lambda)\mathbf{R}_2)^2\right] \le \lambda\text{Trace}\left[\mathbf{R}_1^2\right] + (1-\lambda)\text{Trace}\left[\mathbf{R}_2^2\right] \tag{81}$$

by expanding

$$\lambda^2\text{Trace}\left[\mathbf{R}_1^2\right] + (1-\lambda)^2\text{Trace}\left[\mathbf{R}_2^2\right] + 2\lambda(1-\lambda)\text{Trace}\left[\mathbf{R}_1\mathbf{R}_2\right] \le \lambda\text{Trace}\left[\mathbf{R}_1^2\right] + (1-\lambda)\text{Trace}\left[\mathbf{R}_2^2\right] \tag{82}$$

and rearranging

$$-\lambda(1-\lambda)\text{Trace}\left[\mathbf{R}_1^2\right] + (1-\lambda)(1-\lambda-1)\text{Trace}\left[\mathbf{R}_2^2\right] + 2\lambda(1-\lambda)\text{Trace}\left[\mathbf{R}_1\mathbf{R}_2\right] \le 0 \tag{83}$$

[4] We leave the last step nonspecific since it has been previously shown that a variety of update sequences lead to optimal results [1].

and factoring to obtain

$$-\lambda(1-\lambda)\left[\text{Trace}\left[\mathbf{R}_1^2\right]+\text{Trace}\left[\mathbf{R}_2^2\right]-2\text{Trace}\left[\mathbf{R}_1\mathbf{R}_2\right]\right]=-\lambda(1-\lambda)\left[\text{Trace}\left[(\mathbf{R}_1-\mathbf{R}_2)^2\right]\right]\leq 0 \tag{84}$$

which is clearly true, with equality **iff** $\lambda = 0, 1$.

Thus, simple adaptation of results for sum capacity in [10, 11] shows that another type of simultaneous "waterfilling" (equation (80)) *is* the optimal solution for TSC as well. Therefore, since interference avoidance applied to TSC produces simultaneously "waterfilled" ensembles, interference avoidance is guaranteed to produce codeword ensembles which absolutely minimize TSC. **MAKE SURE THIS IS TRUE!!!!! Check Wei's paper CAREFULLY. Convexity of the metric should be enough.**

Of course, the algorithm is not guaranteed to render $\mathbf{R}$ as a set of scaled identity blocks since it may be impossible to do so. For example, consider a single user with

$$\begin{bmatrix}\mathbf{I}\\ \vdots\\ \mathbf{I}\end{bmatrix}\mathbf{S}\mathbf{S}^\top\begin{bmatrix}\mathbf{I} & \cdots & \mathbf{I}\end{bmatrix}+\mathbf{W}=\begin{bmatrix}\mathbf{S}\mathbf{S}^\top & \cdots & \mathbf{S}\mathbf{S}^\top\\ \vdots & \ddots & \vdots\\ \mathbf{S}\mathbf{S}^\top & \cdots & \mathbf{S}\mathbf{S}^\top\end{bmatrix}+\begin{bmatrix}\mathbf{W}_1 & & \\ & \ddots & \\ & & \mathbf{W}_B\end{bmatrix} \tag{85}$$

and non-white $\mathbf{W}_j$. However, when it *is* possible to satisfy the conditions of Theorem 7 then minimizing TSC is equivalent to maximizing sum capacity and interference avoidance will produce an optimum codeword ensemble.

4 Discussion and Conclusions

Under an assumption of flat channels between users and multiple collaborating bases, we have derived easily computable bounds on sum capacity. In addition, assuming that there exists a codeword ensemble which renders the received covariance matrix as a set of $N \times N$ scaled identity matrix sub-blocks, we have derived a simplified interference avoidance algorithm based on total squared corellation (TSC). Whereas the complexity of each interference avoidance step in the general multiple base case [1] is on the order of $(NB)^3$ owing to the necessary inversion of the received covariance matrix $\mathbf{R}$, for flat channels, the complexity is on the order of $(NB)^2 + N^3$ where the first term is the complexity of the subspace projection from step 3 of the interference avoidance algorithm, and the second term is the complexity of finding the minimum eigenvector in step 4.

Numerical tests for a variety of dimensions and numbers of users and bases show that the sum capacity improvement afforded by interference avoidance is generally not as large as for non-flat gains [1]. For example, with $N = 6$ signal space dimension, $B = 4$ bases and 20 users, the improvement over randomly chosen codewords was on average 12% as compared to an average of 30% for non-flat channels. However, a secondary feature of optimal codeword ensembles is to make user codeword SINRs uniform – which could be useful for integrated receiver structures. That is, each "rail" associated with each codeword would receive the same power, see the same type of interference and have the same SINR. This uniformity plays to the inherent parallelism of integration.

The uniformity of SINRs for codewords and previous results for single receiver systems where interference avoidance maximized user capacity [5] naturally raised the issue making SINRs uniform across all users. Here the ultimate SINRs users receive is preordained by the values of the $\{\mathbf{g}_k\}$ – and this suggested an amusing/interesting sub-problem which arises at the fixed points produced by interference avoidance. Specifically, assume that the optimum ensemble always produces $\mathbf{R}_{ij} = \frac{\text{Trace}[\mathbf{R}_{ij}]}{N}\mathbf{I}$. Using the definition $E_{ij} = \text{Trace}\left[\mathbf{R}_{ij}\right]$ and $\mathbf{E}$ the matrix whose elements are E_{ij} we will then have

$$\mathbf{R} = \mathbf{E} \otimes \mathbf{I} \tag{86}$$

With γ_k defined as the SINR experienced by any codeword i of user k, $\mathbf{s}_{ki}$, we then have (assuming MMSE filtering [1])

$$\mathbf{G}_k^\top \mathbf{R}^{-1} \mathbf{G}_k \mathbf{s}_{ki} = \frac{\gamma_k}{1+\gamma_k}\mathbf{s}_{ki} \tag{87}$$

We then note that [4]

$$\mathbf{R}^{-1} = \mathbf{E}^{-1} \otimes \mathbf{I} \tag{88}$$

Furthermore, if we define $\boldsymbol{\mathcal{W}}$ as the $B \times B$ matrix with elements $\text{Trace}\left[\mathbf{W}_{ij}\right]$, and $P = \text{Trace}\left[\mathbf{S}_k \mathbf{S}_k^\top\right]$ $\forall k$, then

$$\mathbf{E} = P \sum_{\ell} \mathbf{g}_\ell \mathbf{g}_\ell^\top + \boldsymbol{\mathcal{W}} \tag{89}$$

Finally we note that

$$\mathbf{G}_k = \mathbf{g}_k \otimes \mathbf{I} \tag{90}$$

so that all told, for any codeword $\mathbf{s}_{ki}$ we have

$$\mathbf{s}_{ki}^\top \mathbf{G}_k^\top \mathbf{R}^{-1} \mathbf{G}_k \mathbf{s}_{ki} = \mathbf{g}_k^\top \mathbf{E}^{-1} \mathbf{g}_k = \frac{\gamma_k}{1+\gamma_k} \tag{91}$$

Now, what is interesting about equation (91) is that it is a standard interference avoidance problem owing to the definition of $\mathbf{E}$ in equation (89). That is, one could imagine iteratively adjusting the $\mathbf{g}_k$ to greedily maximize γ_k; i.e., make $\mathbf{g}_k$ a maximum eigenvector of $(\mathbf{E} - P\mathbf{g}_k\mathbf{g}_k^\top)^{-1}$. If we assume $|\mathbf{g}_i| = |\mathbf{g}_j| = g\ \forall i,j$ then the end result would be uniform SINR for all users since we could[5] have

$$\mathbf{E} = \frac{E_{\text{tot}}}{B}\mathbf{I} \tag{92}$$

where E_{tot} is the total energy incident on the receivers

$$E_{\text{tot}} = \sum_\ell P \text{Trace}\left[\mathbf{g}_\ell \mathbf{g}_\ell^\top\right] + \text{Trace}\left[\boldsymbol{\mathcal{W}}\right] \tag{93}$$

and B is the number of receivers. Therefore we would have

$$\mathbf{g}_k^\top \mathbf{E}^{-1} \mathbf{g}_k = g^2 \frac{B}{E_{\text{tot}}} \tag{94}$$

and then

$$\gamma_k = \gamma = \frac{Bg^2}{E_{\text{tot}} - Bg^2} \tag{95}$$

This would constitute the attainment of a sort of *user capacity* [6] for the system.

An amusing (and infeasible) way to adjust the $\mathbf{g}_k$ might be to perform a sort of *spatial* interference avoidance where users changed positions to achieve the optimum $\mathbf{g}_k$. Of course, producing arbitrary $\mathbf{g}_k$ would generally be infeasible with the three degrees of freedom afforded by user position. A more practical (and feasible) idea would be to use moderately directive antennas where the modulated waveform could be split according to the optimal $\mathbf{g}_k$ and steered independently toward different bases – a sort of simplistic multi-antenna array.

[5] Assuming enough received signal energy and at least B users so that $P\sum_\ell \mathbf{g}_\ell \mathbf{g}_\ell^\top$ could be chosen to make $P\sum_\ell \mathbf{g}_\ell \mathbf{g}_\ell^\top + \boldsymbol{\mathcal{W}} \propto \mathbf{I}$.

Of course, if such splitting were possible, then it would also be possible to send different signals to different bases resulting in a received signal model of

$$\mathbf{r} = \sum_{ki} \mathbf{s}'_{ki} + \mathbf{w} \tag{96}$$

where each $\mathbf{s}'_{ki}$ is an NB-dimensional vector constrained only in its norm (i.e., a received power constraint). Note that equation (96) is different from the constrained model

$$\mathbf{r} = \sum_{ki} \mathbf{G}_k \mathbf{s}_{ki} + \mathbf{w} \tag{97}$$

used in this paper. Under the model of equation (96) the only constraint is on Trace $[\mathbf{R}]$ and not on the sub-blocks so $\mathbf{R}$ could be rendered (via interference avoidance or any other optimal codeword generation algorithm) as a scaled identity matrix thereby resulting in a greater sum capacity than that achievable under sub-block trace constraints.

Of course, the complexity of such a general procedure would be on the order of $(NB)^3$. In contrast, energy steering alone has complexity B^3 and would be decoupled from codeword adaptation. Thus, the total complexity of energy steering and codeword adaptation would be on the order of $B^3 + (NB)^2 + N^3$, a substantial reduction over $(NB)^3$. So, we close with this notion of "spatial interference avoidance" (or "gross energy steering interference avoidance"), an interesting curiosity which may (or may not) be useful under practical constraints on transceiver complexity.

A Incorporating Carrier Phase Delays

We have thus far assumed complete synchronization at all receivers between all users. Although in baseband such an assumption can be justified through sufficiently long frame durations relative the communications bandwidth allotted, simple propagation delay can cause signals modulated on (say) the in-phase rail to appear on the quadrature rail at the receiver. And although all these relative phases can be compensated for a single user, compensation for multiple users with different delays to the same receivers is not possible in general for omnidirectional transmission.[6]

We therefore introduce *carrier rotation matrices* to cover such cases, and although this complicates the problem slightly, the same types of structural results observed in the synchronized

[6] If broad beams can be directed independently from transmitters to bases (as in section 4) then phase COULD be adjusted by each transmitter so that all are properly syncrhonized at each receivers.

problem still apply because the covariance will still have a fixed trace constraint under codeword variation.

To proceed, we first assume a set of baseband complete orthonormal waveforms in some allotted bandwidth, $\{\phi_i(t)\}$, $i = 1, 2, \cdots, N/2$ where for simplicity and conformance with our previous calculations we assume N even. We then assume a modulation with both $\sin \omega_c t$ and $\cos \omega_c t$ which provides for N passband orthonormal basis functions. Then we represent the i^{th} signature waveform of user ℓ as

$$s_{\ell i}(t) = \begin{bmatrix} s_{\ell i}^{(1)} \phi_1(t) \cos \omega_c t \\ s_{\ell i}^{(2)} \phi_1(t) \sin \omega_c t \\ s_{\ell i}^{(3)} \phi_2(t) \cos \omega_c t \\ s_{\ell i}^{(4)} \phi_2(t) \sin \omega_c t \\ \vdots \\ s_{\ell i}^{(N)} \phi_{\frac{N}{2}}(t) \sin \omega_c t \end{bmatrix} \rightarrow \mathbf{s}_{\ell i} = \begin{bmatrix} s_{\ell i}^{(1)} \\ s_{\ell i}^{(2)} \\ s_{\ell i}^{(3)} \\ s_{\ell i}^{(4)} \\ \vdots \\ s_{\ell i}^{(N)} \end{bmatrix} \tag{98}$$

an N-dimensional codeword.

We then assume that for typical propagation delays τ we have $\phi_i(t) \approx \phi_i(t - \tau)$ for $i = 1, 2, \cdots, N/2$, but that $\cos \omega_c(t-\tau) = \cos \omega_c \tau \cos \omega_c t + \sin \omega_c \tau \sin \omega_c t$ and similarly for the quadrature rail in $\sin \omega_c t$. That is, the propagation delay causes $\phi_i(t) \cos \omega_c t$ to appear as $\alpha \phi_i(t) \cos \omega_c t \pm \sqrt{1 - \alpha^2} \phi_i(t) \sin \omega_c t$ where $-1 \leq \alpha \leq 1$. We therefore define the 2×2 rotation matrix as

$$\mathbf{O}(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \tag{99}$$

and note that the received signal vector after propagation delay τ is

$$\begin{bmatrix} \mathbf{O}(\theta) & & & \\ & \mathbf{O}(\theta) & & \\ & & \ddots & \\ & & & \mathbf{O}(\theta) \end{bmatrix} \mathbf{s}_{\ell i} = \mathbf{\Omega}(\theta) \mathbf{s}_{\ell i} \tag{100}$$

where $\theta = -\omega_c \tau$. The effect of propagation delay is therefore to pairwise rotate signal space components. We also note that $\mathbf{O}^{-1}(\theta) = \mathbf{O}(\theta)^\top = \mathbf{O}(-\theta)$

If we then define the carrier phase rotation from user ℓ to receiver j as $\theta_{\ell j}$ and thence $\mathbf{\Omega}_{\ell j} = \mathbf{\Omega}(\theta_{\ell j})$ as the corresponding $N \times N$ rotation matrix , we can rewrite the received covariance as $\mathbf{R} = \{\mathbf{R}_{ij}\}$, where

$$\mathbf{R}_{ij} = \mathbf{Q}_{ij} + \mathbf{W}_{ij} = \sum_{\ell} g_{\ell i} g_{\ell j} \mathbf{\Omega}_{\ell i} \mathbf{S}_\ell \mathbf{S}_\ell^\top \mathbf{\Omega}_{\ell j}^\top + \mathbf{W}_{ij} \tag{101}$$

The rotation matrix associated with a given user can also be defined as

$$\mathbf{\Omega}_\ell = \begin{bmatrix} \mathbf{\Omega}_{\ell 1} & & & \\ & \mathbf{\Omega}_{\ell 2} & & \\ & & \ddots & \\ & & & \mathbf{\Omega}_{\ell B} \end{bmatrix} \tag{102}$$

and using the gain matrices associated with each user defined in equation (1) we can write the received covariance matrix compactly as

$$\mathbf{R} = \mathbf{Q} + \mathbf{W} = \sum_{\ell} \boldsymbol{\mathcal{G}} \mathbf{S}_\ell \mathbf{S}_\ell^\top \boldsymbol{\mathcal{G}}^\top + \mathbf{W} \tag{103}$$

where

$$\boldsymbol{\mathcal{G}} = \mathbf{\Omega}_\ell \mathbf{G}_\ell \tag{104}$$

Once again we seek to maximize $|\mathbf{R}|$ or minimize Trace $[\mathbf{R}^2]$.

The following lemma will allow us to provide simple bounds for $|\mathbf{R}|$ using Theorem 1 and for Trace $[\mathbf{R}^2]$ using Theorem 6.

Lemma 1 *Let* $\mathbf{A}$ *be a* 2×2 *symmetric matrix*

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Then

$$\textit{Trace}\,[\mathbf{A}\mathbf{O}(\theta)] = \textit{Trace}\,[\mathbf{O}(\theta)\mathbf{A}] = \textit{Trace}\,[\mathbf{A}] \cos\theta$$

Using the definition of $\mathbf{O}(\theta)$, the proof is trivial.

Now, consider $\mathbf{\Omega}_{\ell i} \mathbf{S}_\ell \mathbf{S}_\ell^\top \mathbf{\Omega}_{\ell j}^\top$. We have following Lemma 1

$$\text{Trace}\left[\mathbf{\Omega}_{\ell i} \mathbf{S}_\ell \mathbf{S}_\ell^\top \mathbf{\Omega}_{\ell j}^\top\right] = \text{Trace}\left[\mathbf{S}_\ell \mathbf{S}_\ell^\top \mathbf{\Omega}_{\ell j}^\top \mathbf{\Omega}_{\ell i}\right] = \text{Trace}\left[\mathbf{S}_\ell \mathbf{S}_\ell^\top \mathbf{O}(\theta_{\ell i} - \theta_{\ell j})\right] = M_\ell \cos(\theta_{\ell i} - \theta_{\ell j}) \tag{105}$$

Therefore, we have

Theorem 8 *For flat channels with constant phase rotation, the sub-block traces or $\mathbf{R}$ are*

$$\textit{Trace}\,[\mathbf{R}_{ij}] = E_{ij} = \sum_{\ell} g_{\ell i} g_{\ell j} M_{\ell} \cos(\theta_{\ell i} - \theta_{\ell j}) + \textit{Trace}\,[\mathbf{W}_{ij}] \tag{106}$$

So, flat channels with phase rotation still have covariance matrices with sub-block trace constraints. Therefore, all the bounds derived on $|\mathbf{R}|$ and Trace $[\mathbf{R}^2]$ for constant sub-block traces hold under carrier phase rotation. And in cases where there exist codeword ensembles which can render $\mathbf{R}$ as scaled identity blocks, the bounds will once again be met with equality.

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