Asymptotic Analysis of Wireless Communications Systems:

Which way to infinity?

Michael L. Honig

Department of Electrical & Computer Engineering

Northwestern University

Evanston, IL

Asymptotic Analysis

- Coding theorems (Shannon)
  - Block length $\to \infty$
- Discrete adaptive estimation
  - Continuous-time limit
- Stochastic systems
  - Fluid limits, heavy traffic limits

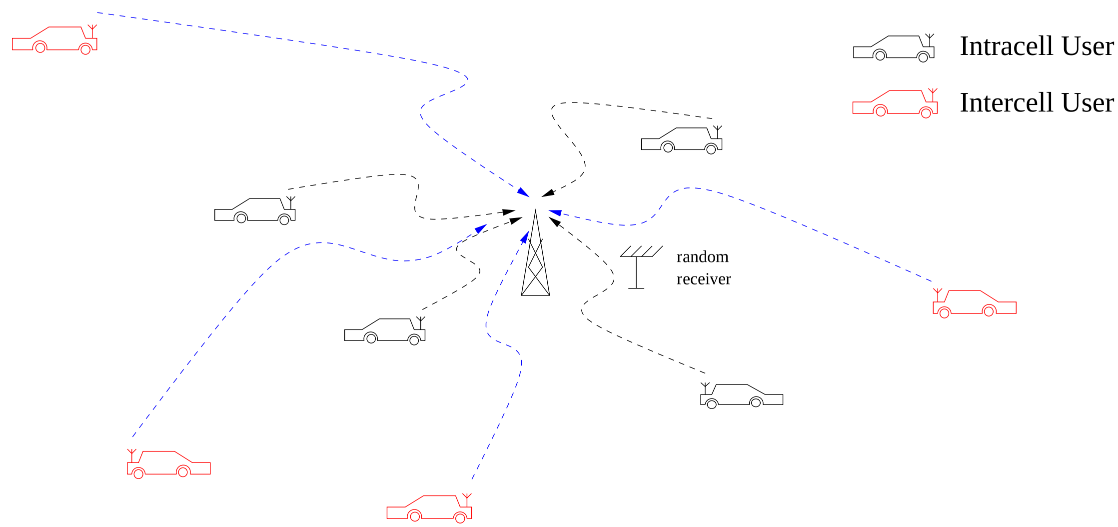
Wireless System Parameters

- Degrees of freedom
  - Bandwidth
  - Dimension of signal subspace
  - Antennas
  - Signatures in space/time/frequency
- Users
- Training interval (adaptive estimation)
- Power
- Mathematical tools: random matrix theory, extreme value theory

Outline

- Analysis of Multi-carrier (MC)-Code-Division Multiple Access
  - Joint work with Matthew Peacock, Iain Collings
    University of Sydney
- Signature optimization with limited feedback
  - Joint work with Wiroonsak Santipach
- Convergence analysis of adaptive Least Squares estimators
  - Joint work with Weimin Xiao

Reverse-Link Model: Code-Division Multiple Access



Received $N \times 1$ vector: $\mathbf{r} = \sum_{k=1}^{K} \sqrt{P_k} b_k \mathbf{s}_k + \mathbf{n} = \mathbf{S}\mathbf{P}^{1/2}\mathbf{b} + \mathbf{n}$

- $\mathbf{s}_k$ is the $N \times 1$ random *i.i.d.* signature for user k, $\mathbf{S} = [\mathbf{s}_1, \cdots, \mathbf{s}_K]$
- b_k is the symbol for user k, $\mathbf{b} = [b_1, \cdots, b_K]^T$
- P_k is the power for user k, $\mathbf{P} = \text{diag}[P_1, \cdots, P_K]$
- $\mathbf{n}$ is additive white Gaussian noise, $E[\mathbf{n}\mathbf{n}^\dagger] = \sigma_n^2 \mathbf{I}$

Linear Minimum Mean Squared Error (MMSE) Receiver

Filter $\mathbf{c}$ for user k is selected to minimize $E[|b_k - \mathbf{c}^\dagger \mathbf{r}|^2]$:

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{s}_k \qquad \text{where} \quad \mathbf{R} = \mathbf{S}\mathbf{P}\mathbf{S}^\dagger + \sigma_n^2 \mathbf{I}$$

Output Signal-to-Interference Plus Noise Ratio:

$$\begin{aligned} \text{SINR}_k &= \mathbf{s}_k^\dagger \left(\mathbf{S}_k \mathbf{P} \mathbf{S}_k^\dagger + \sigma_n^2 \mathbf{I} \right)^{-1} \mathbf{s}_k \\ &= \text{trace}\left\{ \mathbf{R}_k^{-1} \mathbf{s}_k \mathbf{s}_k^\dagger \right\} \\ E[\text{SINR}_k | \mathbf{S}_k] &= \frac{1}{N} \text{trace}\{\mathbf{R}_k^{-1}\} \end{aligned}$$

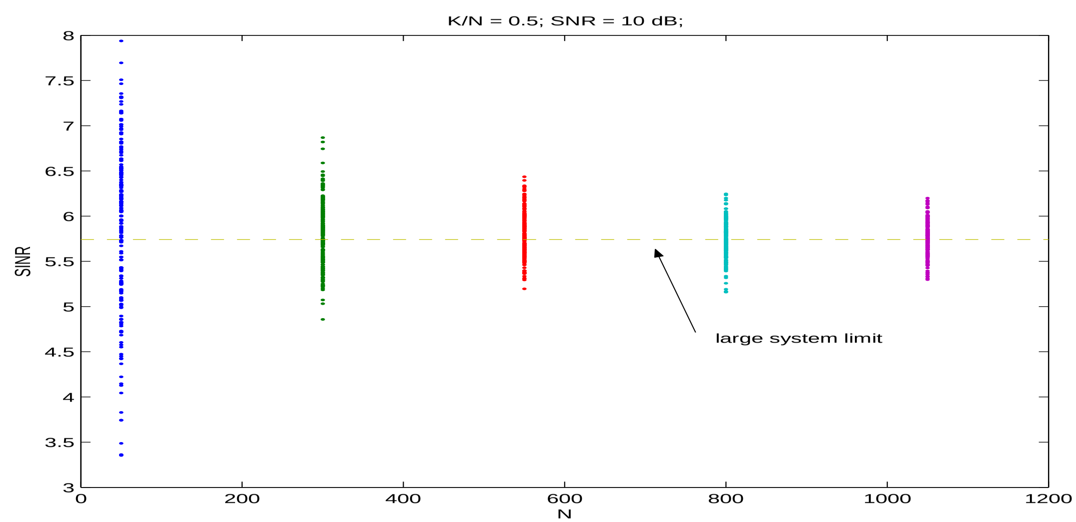
where $\mathbf{S}_k = [\mathbf{s}_1 \cdots \mathbf{s}_{k-1} \mathbf{s}_{k+1} \cdots \mathbf{s}_K]$ and $\mathbf{R}_k = \mathbf{S}_k \mathbf{P} \mathbf{S}_k^\dagger + \sigma_n^2 \mathbf{I}$

Large System SINR (Tse-Hanly)

Users $K \to \infty$; Degrees of freedom $N \to \infty$; $\bar{K} = K/N$ fixed

$$\lim_{(K,N)\to\infty} \text{SINR}_k = \beta_k^\infty = P_k\gamma^\infty, \quad \text{and} \quad \gamma^\infty = \frac{1}{\sigma_n^2 + \bar{K}\int \frac{P dF(P)}{1+\gamma^\infty P}}$$

where $\{P_1, \cdots, P_K\} \xrightarrow{\mathcal{D}} F(\cdot)$

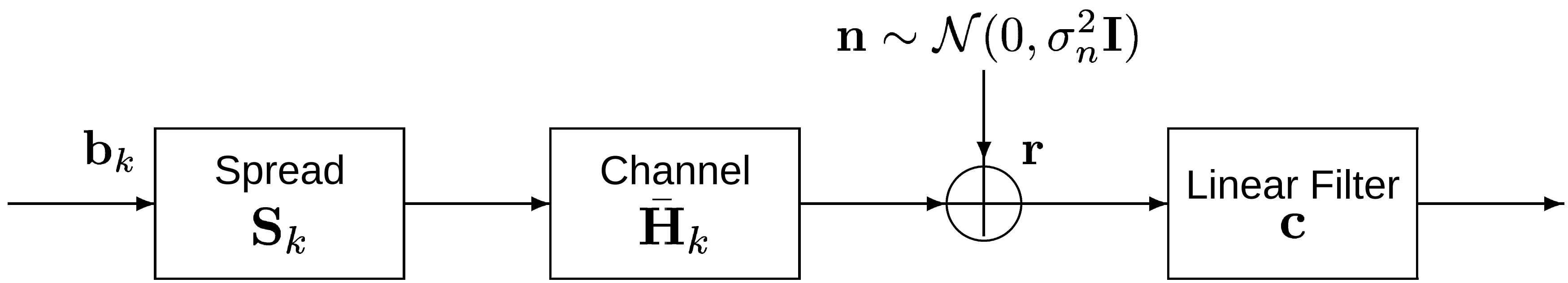


Derivation

$$\gamma^{\infty} = \lim_{N\to\infty} \frac{1}{N} \mathrm{trace}\{\mathbf{R}_k^{-1}\} = \lim_{N\to\infty} (\mathbf{s}_k^{\dagger}\mathbf{R}_k^{-1}\mathbf{s}_k)$$

$$\begin{aligned}
1 &= \lim_{N\to\infty} \frac{1}{N}\text{trace}\,(\mathbf{R}^{-1}\mathbf{R}) \\
&= \lim_{N\to\infty} \text{trace}\,\left[\mathbf{R}^{-1}(\sigma_n^2\mathbf{I} + \mathbf{SPS}^{\dagger})\right] \\
&= \sigma_n^2 \lim_{N\to\infty} \frac{1}{N}\text{trace}\,\mathbf{R}^{-1} + \lim_{N\to\infty} \sum_{k=2}^{N} \frac{1}{N} \frac{P_k \frac{1}{N}\text{trace}\,\mathbf{R}_k^{-1}}{1 + P_k(\mathbf{s}_k^{\dagger}\mathbf{R}_k^{-1}\mathbf{s}_k)} \\
&= \sigma_n^2\gamma^{\infty} + \bar{K}\int \frac{P\gamma^{\infty}}{1 + P\gamma^{\infty}} dF(P)
\end{aligned}$$

Dispersive Channel



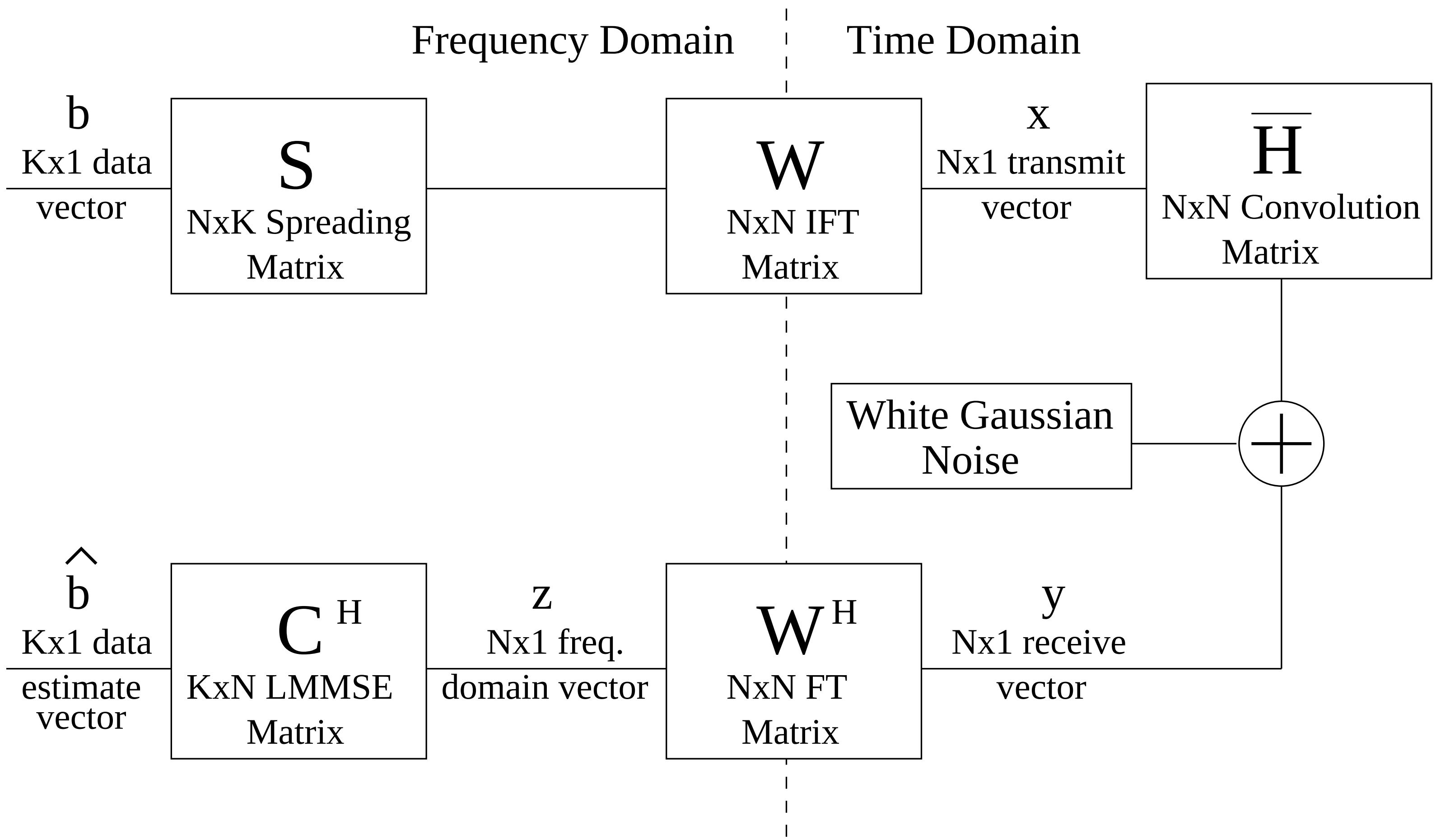
$$\mathbf{r} = \sum_{k=1}^{K} \bar{\mathbf{H}}_k \mathbf{S}_k \mathbf{b}_k + \mathbf{n}$$

where $\bar{\mathbf{H}}_k$, $k = 1, \cdots K$, is assumed to be circulant:

$$\bar{\mathbf{H}}_k = \mathbf{W}\mathbf{H}_k\mathbf{W}^\dagger$$

where $\mathbf{W}$ is the Inverse DFT matrix, and $\mathbf{H}_k$ is diagonal.

Multi-Carrier CDMA



MC-CDMA Model

Received signal (frequency-domain):

$$\mathbf{z} = \sum_{j=1}^{J} \mathbf{H}_j \mathbf{S}_j \mathbf{b}_j + \mathbf{m} = \sum_{j=1}^{J} \sum_{k=1}^{K_j} \mathbf{H}_j \mathbf{s}_{j,k} b_{j,k} + \mathbf{m}$$

- $\mathbf{H}_j$ is diagonal with random *i.i.d* elements. For Rayleigh channels, elements are complex Gaussian.
- User j, $1 \le j \le J$, has signature matrix $\mathbf{S}_j = [\mathbf{s}_{j,1}, \cdots, \mathbf{s}_{j,K_j}]$, $K = \sum_{j=1}^{J} K_j$
- $\mathbf{b}_j$ is a vector of K_j data symbols
- $\mathbf{m} = \mathbf{W}^{\dagger}\mathbf{n}$ is zero-mean AWGN with covariance $\sigma_n^2 \mathbf{I}$.

Large System Limits: MC-CDMA

1. Infinite user limit; single signature per user: $K_j = 1, j = 1, \cdots, J$
   $J \to \infty$, $N \to \infty$, $\bar{J} = J/N$ fixed
   - Li, Tulino, Verdú; Peacock, Collings, Honig

2. Single user; infinite signature limit
   - Li, Tulino, Verdú (downlink); Peacock, Collings, Honig

3. Finite users J; infinite signature limit
   $K_j \to \infty$, $N \to \infty$, $\bar{K}_j = K_j/N$ fixed for each $j = 1, \cdots, J$
   - Can evaluate asymptotic SINR and capacity *regions*

MC-CDMA SINR

Average output SINR of the MMSE filter for the k^{th} signature of user u is

$$\beta_u = \frac{1}{N}\,\mathrm{trace}\left(\mathbf{R}^{-1}\mathbf{P}_u\right)$$

- where $\mathbf{R} = \sigma_n^2 \mathbf{I}_N + \sum_{u=1}^{J} \widetilde{\mathbf{S}}_u \widetilde{\mathbf{S}}_u^{\dagger}$ and $\widetilde{\mathbf{S}}_u = \mathbf{H}_u \mathbf{S}_u$
- $\mathbf{P}_u = \mathbf{H}_u \mathbf{H}_u^{\dagger}$ is the diagonal matrix of powers in each subchannel
- Computing the large system limit is more complicated.

Eigenvalue Distribution

- For the $N \times N$ Hermitian matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_N$, the empirical distribution function (e.d.f.) of the eigenvalues of $\mathbf{A}$ is

$$G_{\mathbf{A}}^N(x) = \frac{1}{N} \cdot |\{\lambda_i : \lambda_i \leq x\}|$$

- The Stieltjes transform of $G_{\mathbf{A}}(x)$ is

$$m_{\mathbf{A}}(z) = \int \frac{1}{\lambda - z} dG_{\mathbf{A}}(\lambda) \qquad \text{for } z \in \mathbb{C}^+$$

- $\lim_{N\to\infty} \frac{1}{N} \operatorname{trace}\{\mathbf{R}^{-1}\} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i} = \int \frac{1}{\lambda} \, dG_{\mathbf{R}}(\lambda)$
$= \lim_{z\to 0} m_{\mathbf{R}_k}(z)$

Key Random Matrix Results

- (Silverstein '95) If $\mathbf{R} = \mathbf{B} + \mathbf{SDS}^\dagger$, then as $N \to \infty$ with $\bar{K} = K/N$, $m_{\mathbf{R}}(z)$ satisifes

$$m_{\mathbf{R}}(z) = m_{\mathbf{B}}\left(z - \bar{K}\int \frac{\lambda}{1+\lambda m_{\mathbf{R}}(z)} dG_{\mathbf{D}}(\lambda)\right)$$

- R-transform of $m_{\mathbf{B}}(z)$: $R_{\mathbf{B}}(y) = \frac{1}{y} + m_{\mathbf{B}}^{-1}(y)$
  If $\mathbf{R}_N = \mathbf{A}_N + \mathbf{B}_N$ and $\mathbf{A}_N$ and $\mathbf{B}_N$ form a free family in the large matrix limit, then $R_{\mathbf{R}}(y) = R_{\mathbf{A}}(y) + R_{\mathbf{B}}(y)$

- S-Transform: If $\mathbf{C}_N = \mathbf{A}_N\mathbf{B}_N$, then $S_{\mathbf{C}}(x) = S_{\mathbf{A}}(x)S_{\mathbf{B}}(x)$

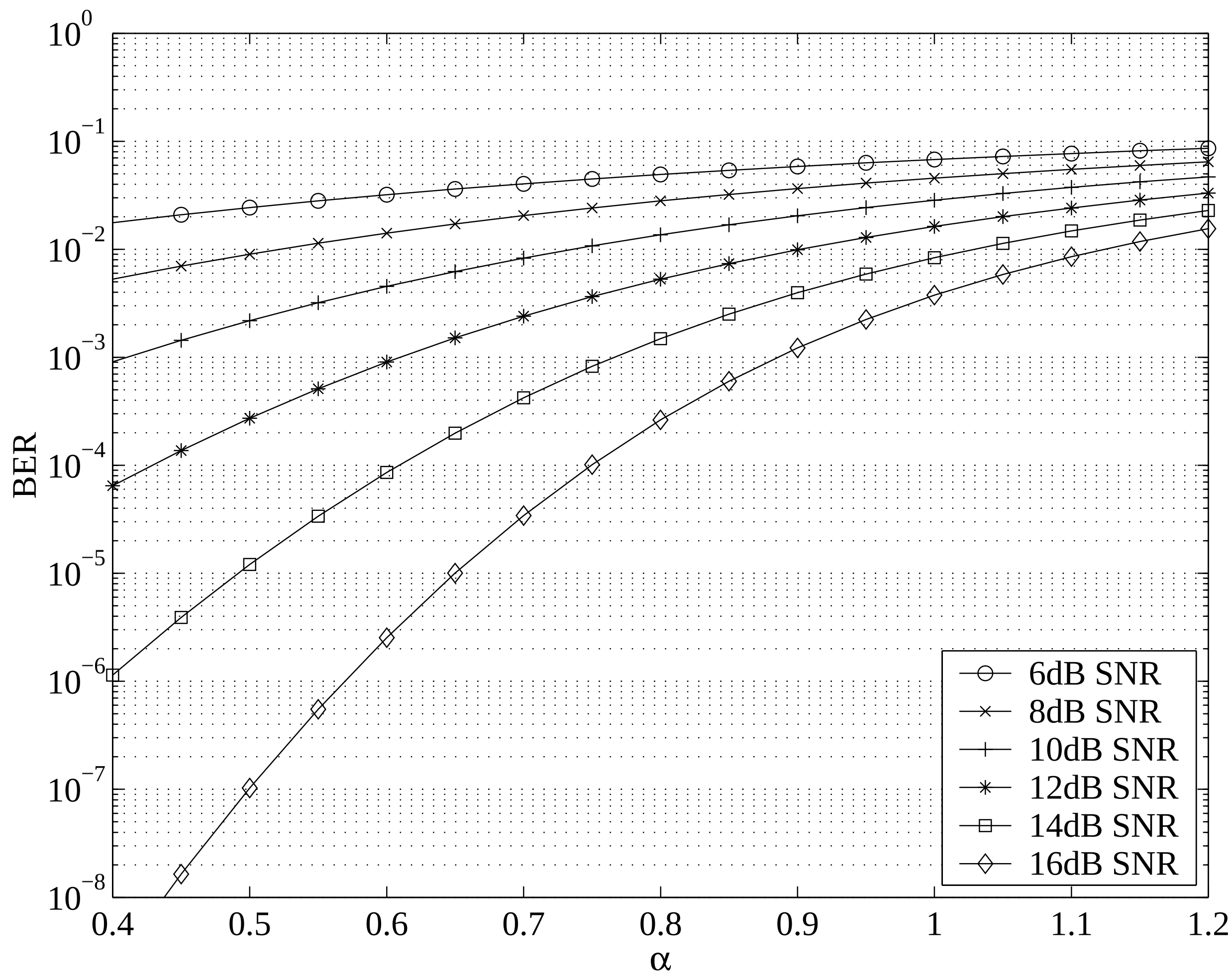
Single-User Infinite-Signature Limit

As $N, K \to \infty$, the SINR at the output of the MC-CDMA LMMSE receiver converges in probability to:

$$\beta^{\infty} = \frac{1+\beta^{\infty}}{\bar{K}} - \kappa \frac{(1+\beta^{\infty})^2}{\bar{K}^2} \exp\left(\kappa \frac{1+\beta^{\infty}}{\bar{K}}\right) \mathrm{Ei}\left(\kappa \frac{1+\beta^{\infty}}{\bar{K}}\right)$$

- $\kappa = \sigma_n^2 / \sigma_H^2$ and σ_H^2 is the variance of the subchannel gains
- $\mathrm{Ei}\left(x\right) = \int_1^{\infty} e^{-xt} t^{-1} \mathrm{d}t$ (exponential integral)

BER vs System Load (BPSK)



Multi-user Multi-Signature MC-CDMA SINR: Equal Loads

As $(N, K_j) \to \infty$, with fixed $\bar{K}_j = \frac{K_j}{N} = \frac{K}{JN}$, $j \in \{1, \dots J\}$, the SINR for the u^{th} user with power P_u satisfies:

$$1 = \sigma_n^2 \gamma_0 + \frac{\bar{K}}{J} \sum_{j=1}^{J} \frac{\beta_u^\infty / P_u}{P_j^{-2} + \beta_u^\infty / P_u}$$

and $\gamma_0 = m_{\mathbf{R}}(0)$ is determined by solving the following set of $2J + 1$ equations:

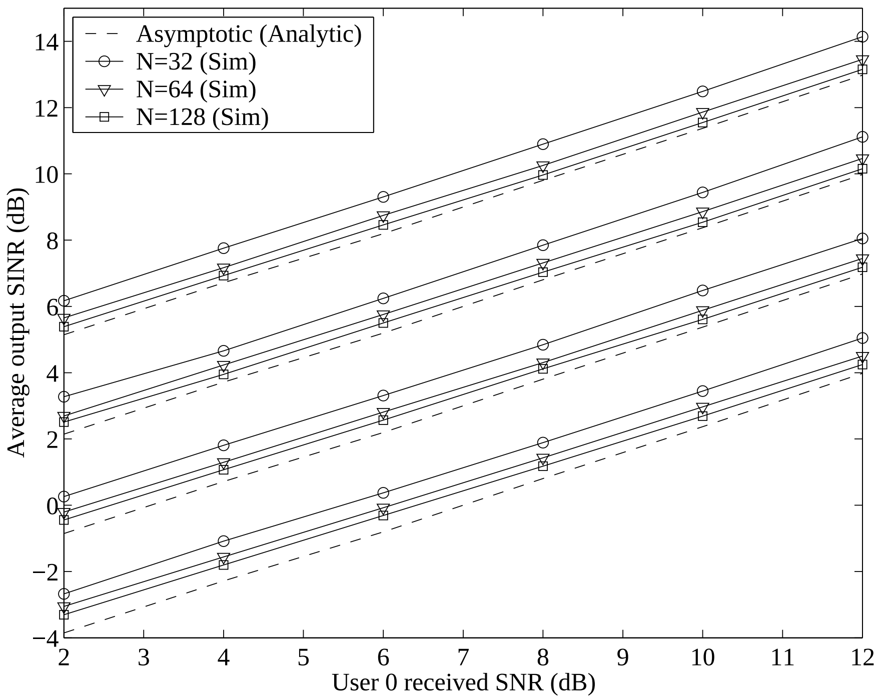
$$\begin{aligned}
\frac{1}{\gamma_0} &= \sigma_n^2 + \frac{J}{\gamma_0} + \sum_{j=1}^{J} z_j \\
\gamma_0 &= \bar{K}_j \gamma_j' - \frac{1}{z_j} + \frac{\bar{K}_j}{z_j} \quad , \quad j = 1 \text{ to } J \\
\frac{1}{\gamma_j'} &= -z_j + \frac{1}{\bar{K}_j \gamma_j'} \left[1 - f\left(\frac{1}{\bar{K}_j P_j^2 \gamma_j'} \right) \right] \quad , \quad j = 1 \text{ to } J \\
f(x) &= x \exp(x) \, \mathrm{Ei}(x)
\end{aligned}$$

Multi-user Multi-signature MC-CDMA SINR: Arbitrary Loads

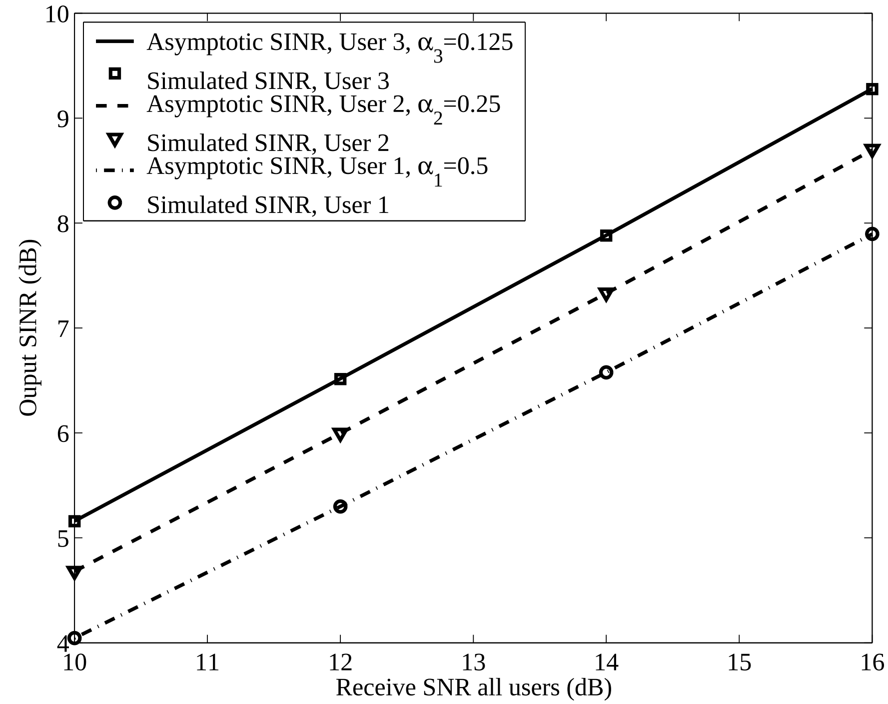
$$\begin{aligned}\frac{1}{\beta_j^{\infty}} &= \lim_{N\to\infty} \frac{1}{N}\,\mathrm{trace}\left(\mathbf{H}_j^{-1}\mathbf{R}\mathbf{H}_j^{-\dagger}\right)\\ &= \mathbf{S}_j\mathbf{S}_j^{\dagger} + \mathbf{H}_j^{-1}\mathbf{R}_j\mathbf{H}_j^{-\dagger}\\ &= \mathbf{T}_1 + \mathbf{T}_2\end{aligned}$$

1. Compute $m_{\mathbf{T}_1}(z)$.
2. Compute S-transform of $\mathbf{T}_2$ in terms of S-transforms of $\mathbf{R}_j$ and $\mathbf{H}_j^{\dagger}\mathbf{H}_j$.
3. Compute $m_{\mathbf{T}_2}(z)$ via inverse S-transform.
4. Compute $m_{\mathbf{T}_1+\mathbf{T}_2}(z)$ via inverse-R transform of $R_{\mathbf{T}_1+\mathbf{T}_2}$.

Asymptotic and Simulated SINRs



(a) Equal No. signatures / Unequal Power



(b) Unequal No. signatures / Equal Power

The user with the least number of signatures has the highest average SINR per signature.

Asymptotic Spectral Efficiency Region

- The capacity region of the J-user MC-CDMA channel is given by [Verdú86]

$$\bigcap_{I\subseteq\{1,\ldots,J\}} \left\{ (\mathsf{C}_1,\cdots,\mathsf{C}_J) \;:\; 0 \le \sum_{i\in I} \mathsf{C}_i \le \frac{1}{N}\log\left|\mathbf{I}_{K^I} + \sigma_n^{-2}\underline{\mathbf{S}}_I^\dagger \underline{\mathbf{S}}_I\right| \right\}$$

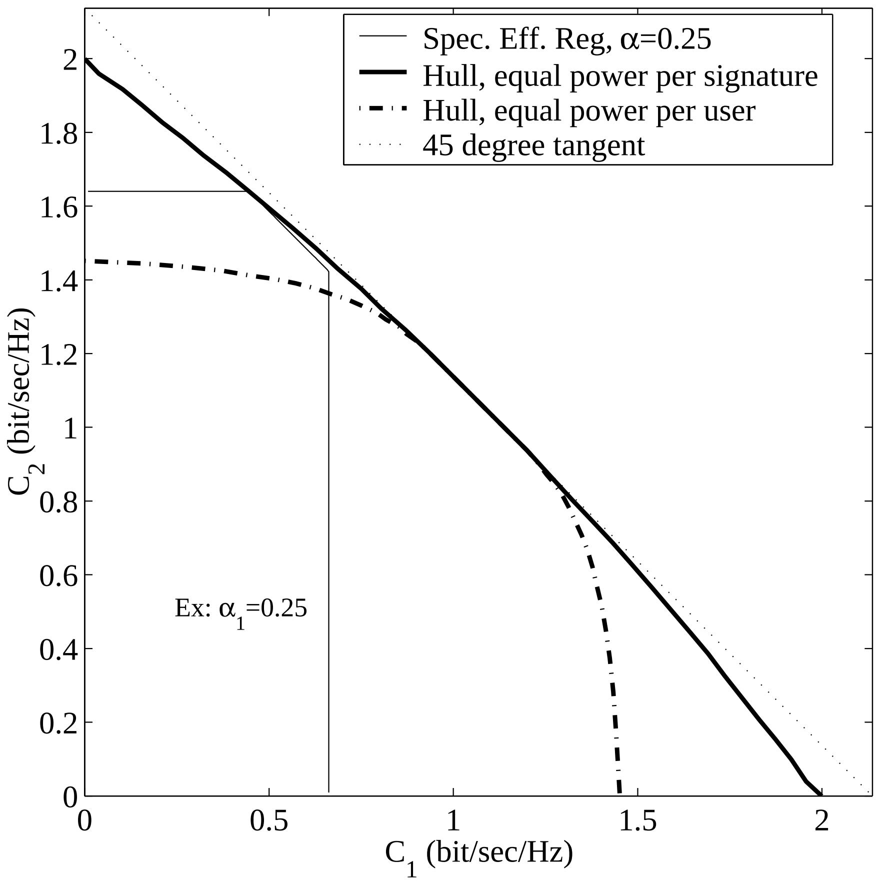
- $K^I = \sum_{i\in I} K_i$ and $\underline{\mathbf{S}}_I = \left[\widetilde{\mathbf{S}}_{I_1} \; \widetilde{\mathbf{S}}_{I_2} \; \ldots \; \widetilde{\mathbf{S}}_{I_{|I|}}\right]$

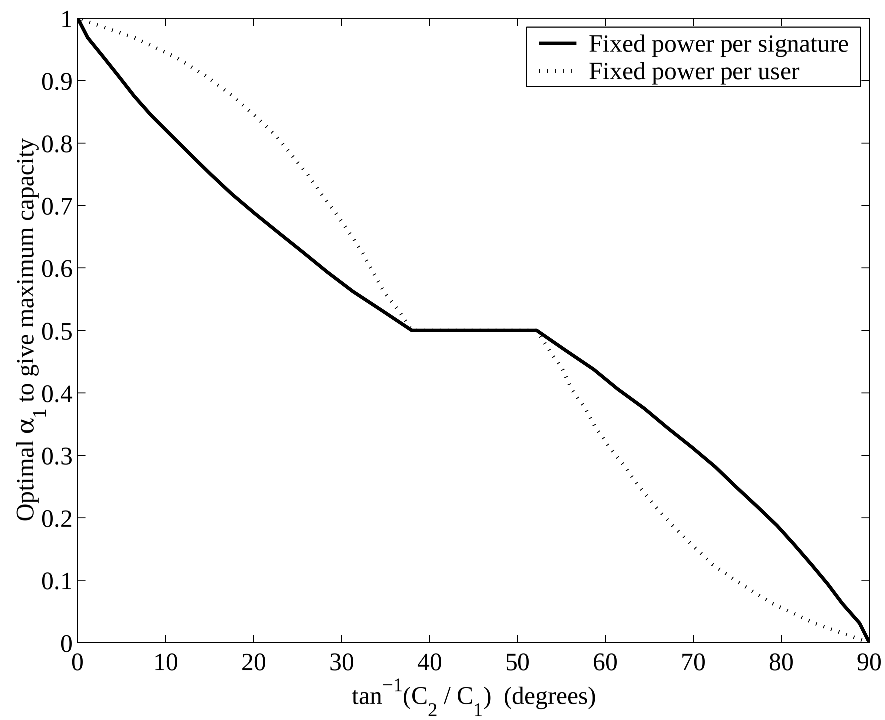
- As $(K_j, N) \to \infty$, $C_i(\sigma_n^2)$ converges to [Müller'02],

$$\mathsf{C}_I^\infty(z) = -\int_{-\infty}^{z} \left(m_{\widetilde{\mathbf{R}}_I}(z') + \frac{1}{z'} \right) \mathrm{d}z'$$

where $z \to \sigma_n^2$, and $m_{\widetilde{\mathbf{R}}_I}(z)$ is the Stieltjes transform of the asymptotic eigenvalue distribution of $\widetilde{\mathbf{R}}_I \triangleq \widetilde{\mathbf{S}}_I^\dagger \widetilde{\mathbf{S}}_I$

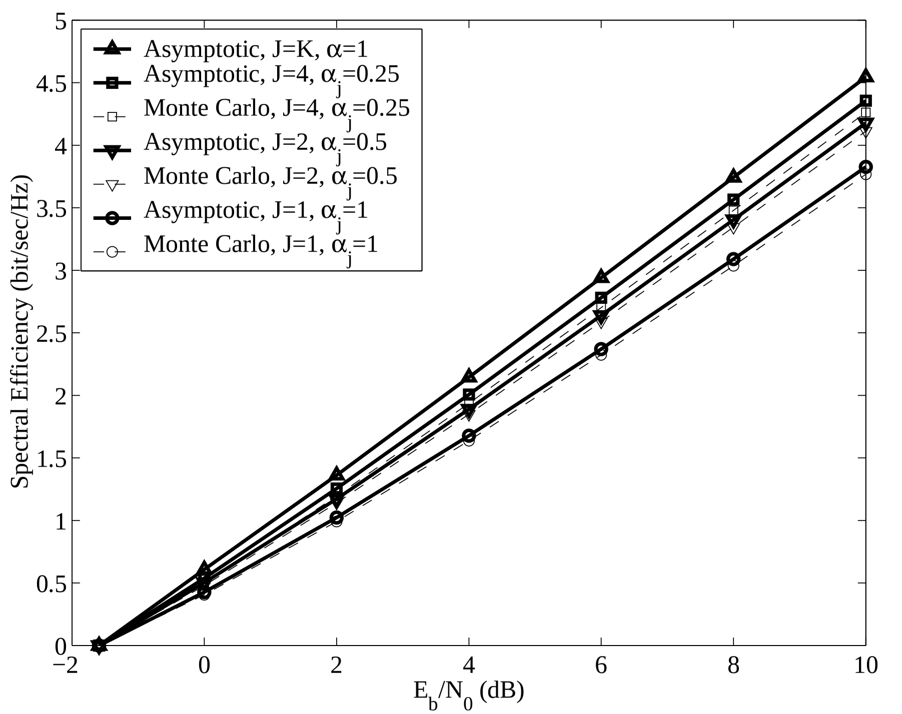
Asymptotic Spectral Efficiency Region: Two Equal-Power Users

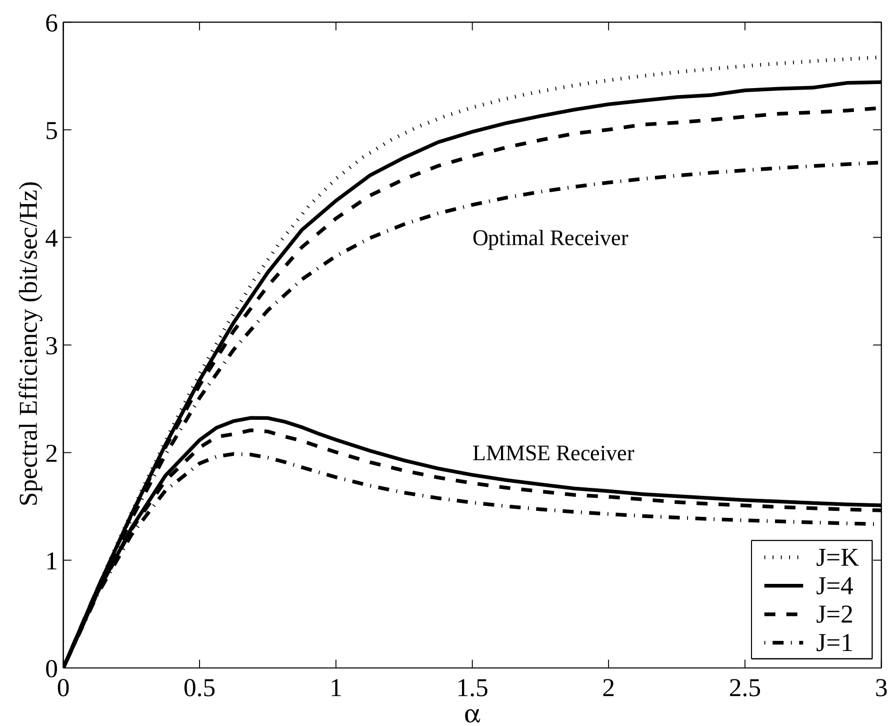




Sum SE is maximized when both users have the same number of signatures.

Asymptotic Sum Spectral Efficiency





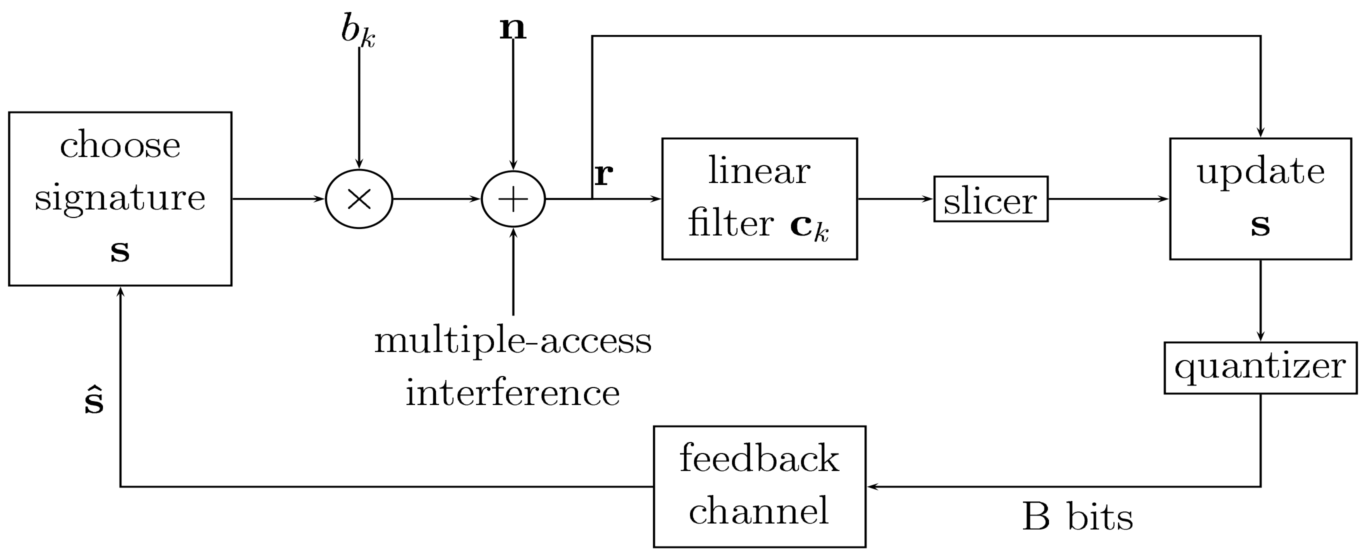
For a fixed load $\bar{K}$, the SE increases with J.

NORTHWESTERN UNIVERSITY

Signature Optimization

- Can optimize transmitted signatures in space/time/frequency to *avoid* interference.
- Prior Work:
  - CDMA: Rapajic & Vucetic (‘95); Jang et al (‘98); Viswanath et al (‘99); Ulukus and Yates (‘99); Rose and Yates (‘99); Popescu and Rose (‘00); Lok and Wong (‘00, ‘01)
  - MIMO & Multiple Access Channels:
    Salz (‘85); Yang & Roy (‘94); Honig & Madhow (‘93);
    Varanasi et al (‘98,‘99); Yu & Cioffi (‘00); Giannakis et al (‘00,‘01)
- AWGN: $\{\mathbf{s}_1, \cdots, \mathbf{s}_K\} \rightarrow$ orthogonal set, $K < N$
- With multipath, optimizing $\mathbf{s}_k$ serves to avoid interference *and* pre-equalize channel.
- Requires feedback from receiver to transmitter.

Single-User Signature Optimization



$$\mathbf{r} = b_k \mathbf{s}_k + \sum_{j \neq k} b_j \mathbf{s}_j + \mathbf{n} \quad (N \times 1)$$

- $\{\mathbf{s}_j\}$, $j \neq k$, are *i.i.d.* and fixed.
- Problem: Select $\mathbf{s}_k$ to maximize SINR $= \beta_k = \frac{|\mathbf{c}^\dagger \mathbf{s}_k|^2}{\mathbf{c}^\dagger \mathbf{R}_k \mathbf{c}}$ subject to $||\mathbf{s}_k||^2 = 1$.

Infinite-Precision Solution ($B \to \infty$) (Ulukus & Yates)

- $\mathbf{s}_k = \kappa \mathbf{c}$, $\kappa = 1/\|\mathbf{c}\|$
- MMSE receiver: $\mathbf{c} = \mathbf{R}^{-1}\mathbf{s}_k \implies \mathbf{s}_k$ is the eigenvector of $\mathbf{R}_k$ corresponding to the minimum eigenvalue.
  - Select $\mathbf{s}_k$ to lie in the direction of minimum interference plus noise.
- $\mathbf{c} = \mathbf{s}_k/\kappa$ (matched filter is the MMSE filter)

Finite-Precision Solution: Vector Quantization

- Set of possible signatures for user k:
  - $\mathcal{V} = \{\mathbf{v}_j\}, 1 \le j \le 2^B$,
  - $\|\mathbf{v}_j\| = 1$ for each j.
- The receiver selects

$$\mathbf{s}_k = \arg\min_{\mathbf{v} \in \mathcal{V}} X_j$$

where $X_j = \sum_{i \neq k} (\mathbf{v}_j^\dagger \mathbf{s}_i)^2$ (total squared interference).

- Average SINR: $\bar{\beta}_k = \frac{1}{\sigma_n^2 + E[I_{\min}]}$ where $I_{\min} = \min\{X_1, \cdots, X_{2^B}\}$.

Large System Limit

Let $(K, N, B) \rightarrow \infty$ with fixed

$$\begin{aligned} \bar{K} &= K/N \\ \bar{B} &= B/N \end{aligned}$$

Asymptotic SINR for user k:

$$\lim_{(K,N,B)\rightarrow\infty} \bar{\beta}_k = \beta_k^{\infty}$$

Random Vector Quantization (RVQ)

- Choose $\mathcal{V} = \{v_1, \ldots, v_{2B}\}$ from a joint distribution $F_{\mathcal{V}}$.

  Let

$$\beta_{\mathcal{V}}^{\infty} = \lim_{(K,N,B)\to\infty} \bar{\beta}_k$$

- RVQ: $\{v_1, \ldots, v_{2B}\}$ are *i.i.d.*

  Let

$$\beta_{\mathrm{RVQ}}^{\infty} = \lim_{(K,N,B)\to\infty} \bar{\beta}_k$$

- **Theorem:** $\beta_{\mathcal{V}}^{\infty} \leq \beta_{\mathrm{RVQ}}^{\infty}$ for any $F_{\mathcal{V}}$.

RVQ Interference Power

- $F_{X,N}(x) = \text{cdf of } X_i$ for finite processing gain N
- The cdf for $I_{\min,N}$ is

$$F_{I_{\min,N}}(x) = 1 - (1 - F_{X,N}(x))^{2^B}$$

- The average interference is

$$E[I_{\min,N}] = \int_0^\infty x 2^B (1 - F_{X,N}(x))^{2^B - 1} f_{X,N}(x)\, dx$$

Large System Interference Power (RVQ)

- As $N \to \infty$,

$$X_j = \sum_{i \neq k} (v_j^\dagger \mathbf{s}_i)^2 \xrightarrow{\mathcal{D}} X$$

where X has a Gamma cdf F_X with mean $\frac{K}{N}$ and variance $\frac{2K}{N^2}$.

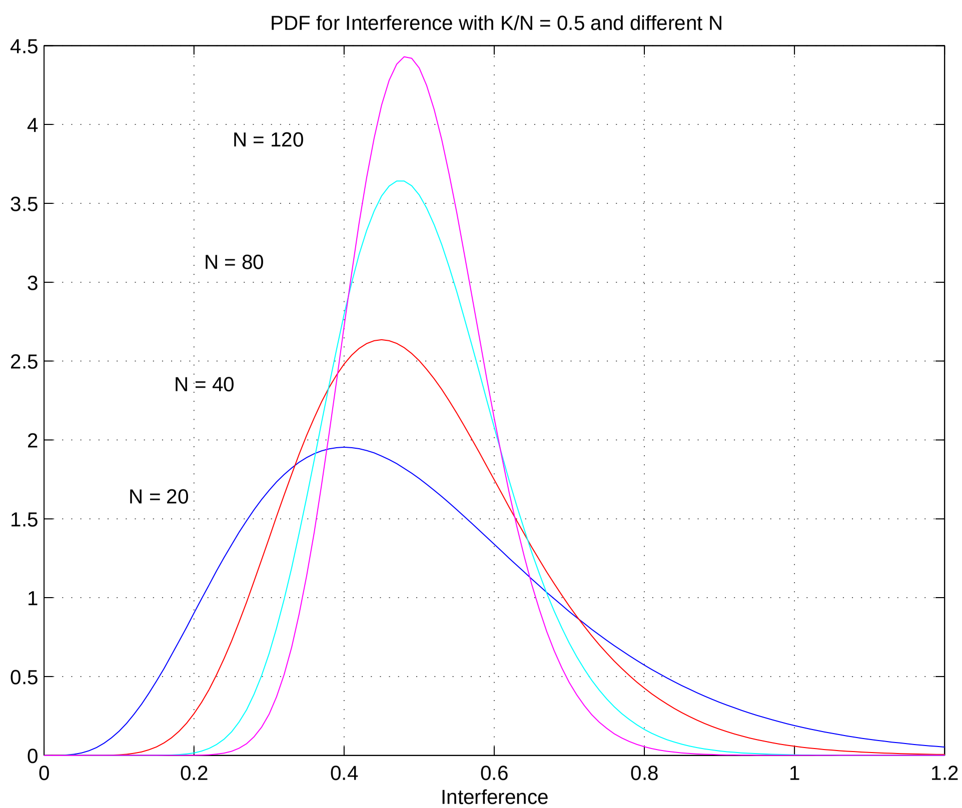
- **Theorem:** The large system interference power is

$$I_{\min}^{\infty} = \lim_{(K,N,B)\to\infty} E\left[I_{\min,N}\right] = \lim_{(K,N,B)\to\infty} F_X^{-1}\left(\frac{1}{2^B}\right)$$

where the latter limit converges for $0 \leq \bar{K} \leq 1$ and $\bar{B} \geq 0$.

Proof: Uses asymptotic theory of extreme statistics.

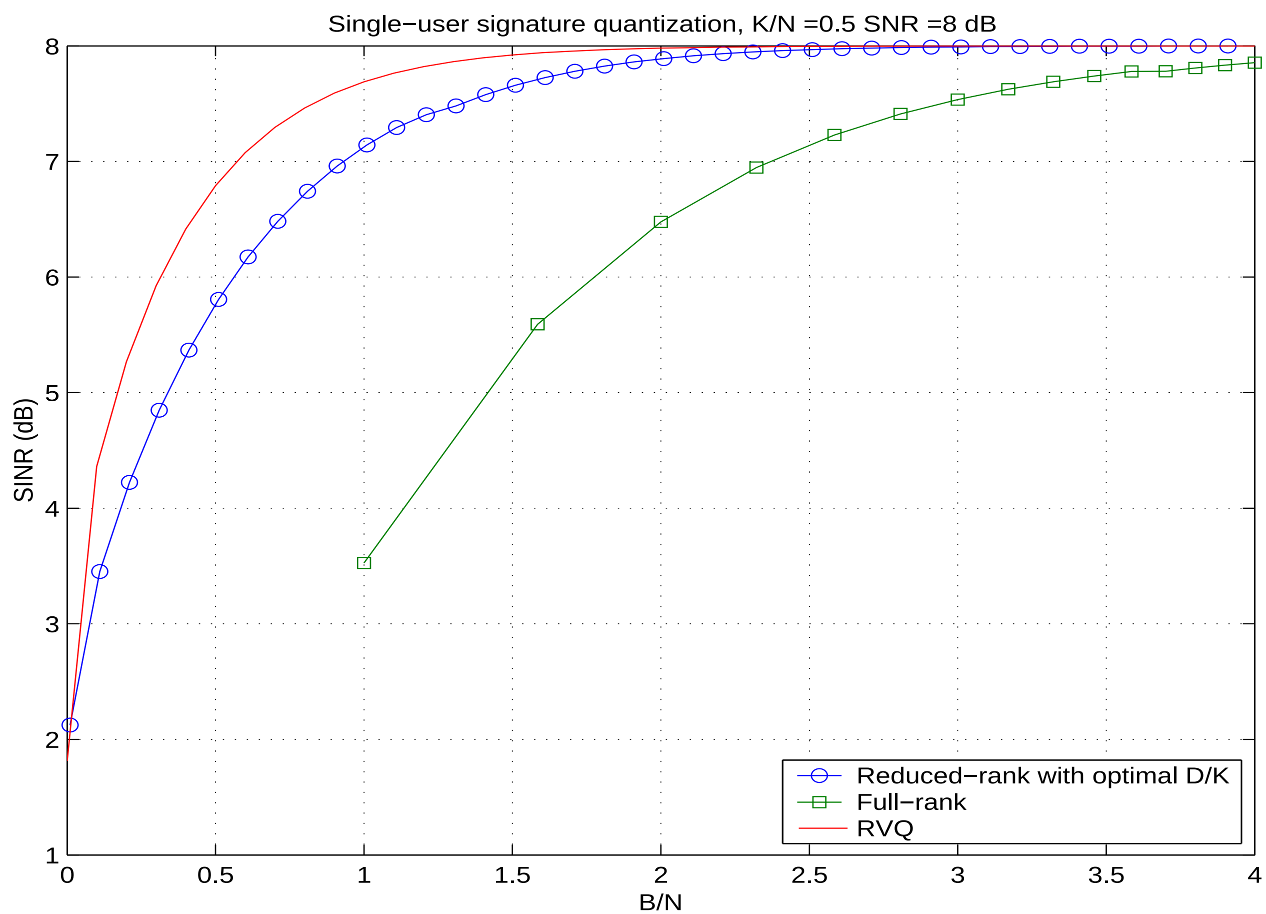
Interference pdf



As $N \to \infty$:

- Number of independent samples $= 2^{\bar{B}N}$
- Variance $\to 0$.
- $E[I_{\min,N}]$ converges to a finite limit.

SINR vs B/N: Single-User Optimization



Reduced-Rank Signature

- Constrain $\mathbf{s}_k$ to lie in D-dimensional subspace:

$$\mathbf{s}_k = \mathbf{F}_k \boldsymbol{\alpha}_k$$

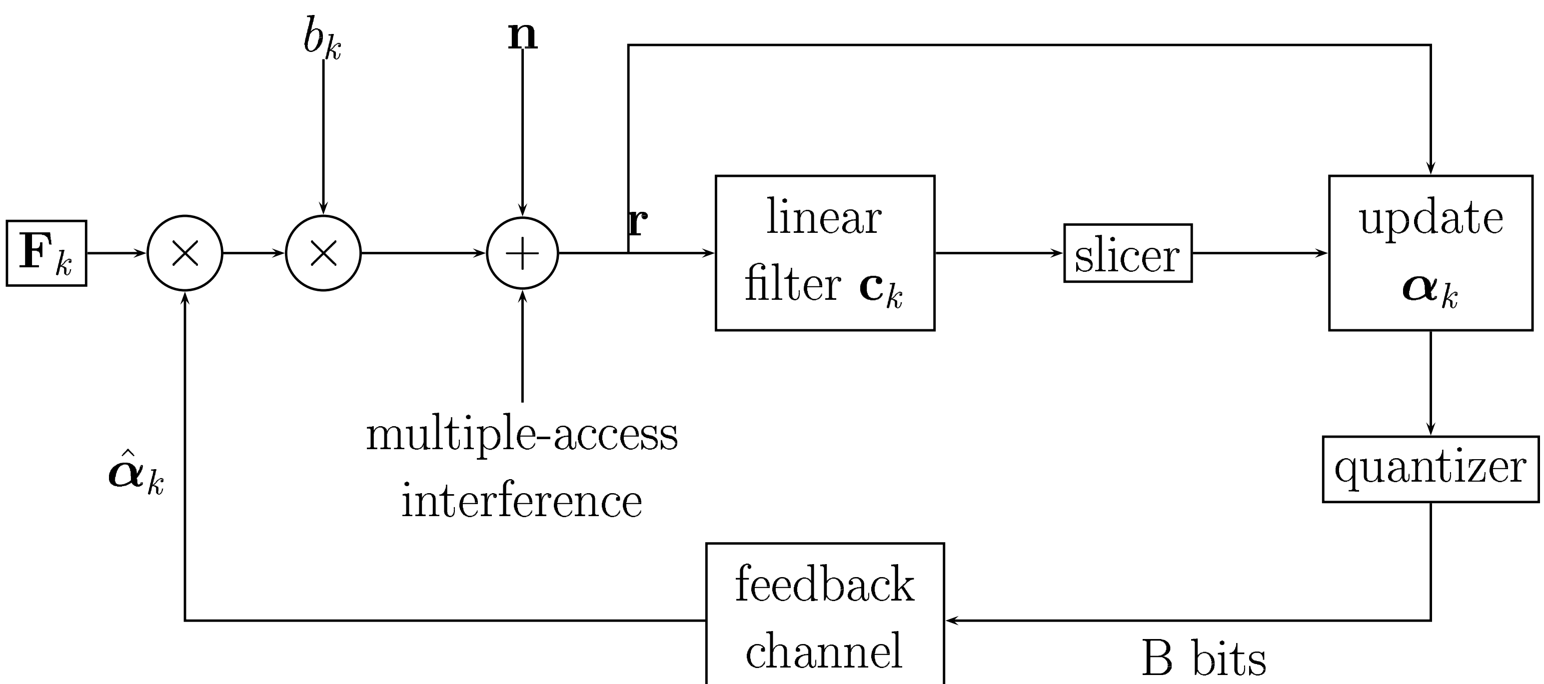
where

$$\begin{aligned} \mathbf{F}_k &= N \times D \text{ Matrix of basis vectors for user } k, \\ \boldsymbol{\alpha}_k &= D \times 1 \text{ Vector of combining coefficients.} \end{aligned}$$

- Example $(\mathbf{F}_k^\dagger \mathbf{F}_k = \mathbf{I})$:

$$\mathbf{F}_k = \begin{bmatrix} \pm 1/\sqrt{2} & 0 \\ \pm 1/\sqrt{2} & 0 \\ 0 & \pm 1/\sqrt{2} \\ 0 & \pm 1/\sqrt{2} \end{bmatrix}$$

System Model: Reduced-Rank Signature



Reduced-Rank Signature Optimization

- Choose $\boldsymbol{\alpha}_k$ to $\min E\left[|b_k - \mathbf{c}_k^\dagger \mathbf{r}|^2\right]$ s.t. $\|\mathbf{s}_k\|^2 = 1$.
- For matched filter ($\mathbf{c}_k = \mathbf{s}_k$),

$$\left[\left(\mathbf{F}_k^\dagger \mathbf{F}_k\right)^{-1} \mathbf{F}_k^\dagger \mathbf{R}_k \mathbf{F}_k\right] \boldsymbol{\alpha}_k = \lambda_{\min} \boldsymbol{\alpha}_k$$

where $\mathbf{R}_k = \mathbf{S}_k \mathbf{S}_k^\dagger + \sigma_n^2 \mathbf{I}$, and $\mathbf{S}_k = [\mathbf{s}_1 \cdots \mathbf{s}_{k-1}\, \mathbf{s}_{k+1} \cdots \mathbf{s}_K]$.

- Quantize $\boldsymbol{\alpha}_k$ and feed back the $D \times 1$ vector $\hat{\boldsymbol{\alpha}}_k$.

Reduced-Rank Trade-off

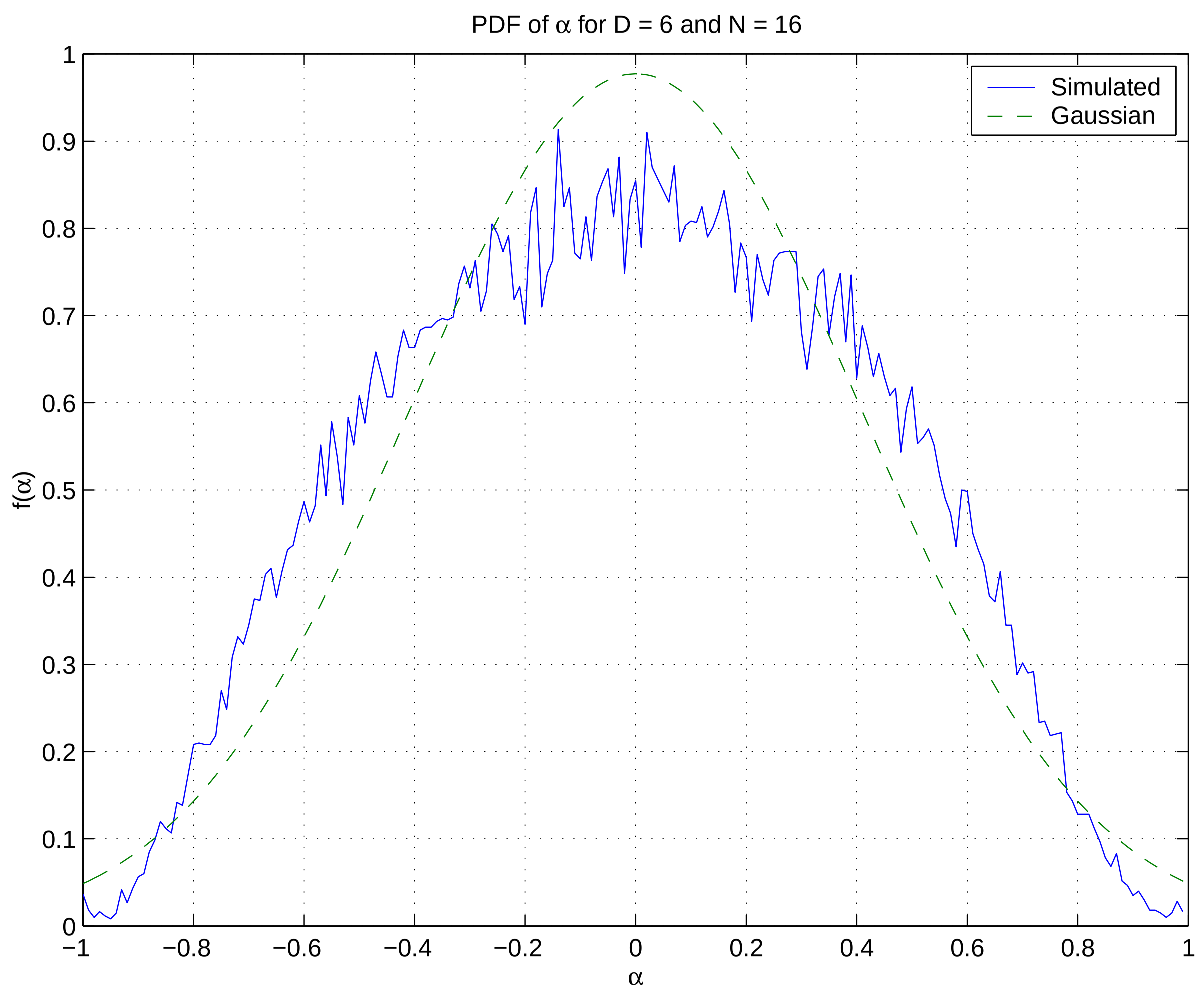
- As D increases:
  - More degrees of freedom for interference avoidance
  - More coefficients to quantize
- For fixed B, what is the optimal D?

Signature Quantization

- Assume:
  - Scalar quantizer with B bits for D coefficients.
  - Elements of $\boldsymbol{\alpha}_k$ are *i.i.d.* with pdf $\mathcal{N}\left(0, \frac{1}{D}\right)$.
  - Error-free feedback channel.
- $\hat{\boldsymbol{\alpha}}_k = \boldsymbol{\alpha}_k + \Delta\boldsymbol{\alpha}_k$, where

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_k &= \text{Quantized vector}, \\ \Delta\boldsymbol{\alpha}_k &= \text{Error vector}. \end{aligned}$$

Example PDF of $\alpha_{k,i}$



Quantization Noise Power

- There are B/D bits or $2^{B/D} = M$ quantization levels for each coefficient.
- For large M and Gaussian pdf,

$$E\left[\|\Delta\boldsymbol{\alpha}_k\|^2\right] = \frac{\sqrt{3}\pi}{2^{2B/D+1}} + O\left(\frac{1}{M^2}\right)$$

SINR with Quantized Signature

- Optimize user k's signature with fixed interference.
- Quantized signature: $\hat{\mathbf{s}}_k = \mathbf{F}_k \hat{\boldsymbol{\alpha}}_k$
- $\mathbf{F}_k$ is random with $\mathbf{F}_k^\dagger \mathbf{F}_k = \mathbf{I}$.
- Averaged SINR is

$$\bar{\beta}_k = \frac{E\left[\hat{\mathbf{s}}_k^\dagger \hat{\mathbf{s}}_k\right]}{E\left[\hat{\mathbf{s}}_k^\dagger \mathbf{R}_k \hat{\mathbf{s}}_k\right]} = \frac{E\left[\|\hat{\boldsymbol{\alpha}}_k\|^2\right]}{\sigma_n^2 E\left[\|\hat{\boldsymbol{\alpha}}_k\|^2\right] + E\left[\hat{\boldsymbol{\alpha}}_k^\dagger \mathbf{F}_k^\dagger \mathbf{S}_k \mathbf{S}_k^\dagger \mathbf{F}_k \hat{\boldsymbol{\alpha}}_k\right]}$$

where $E[\,]$ is over quantization noise.

Large System Limit

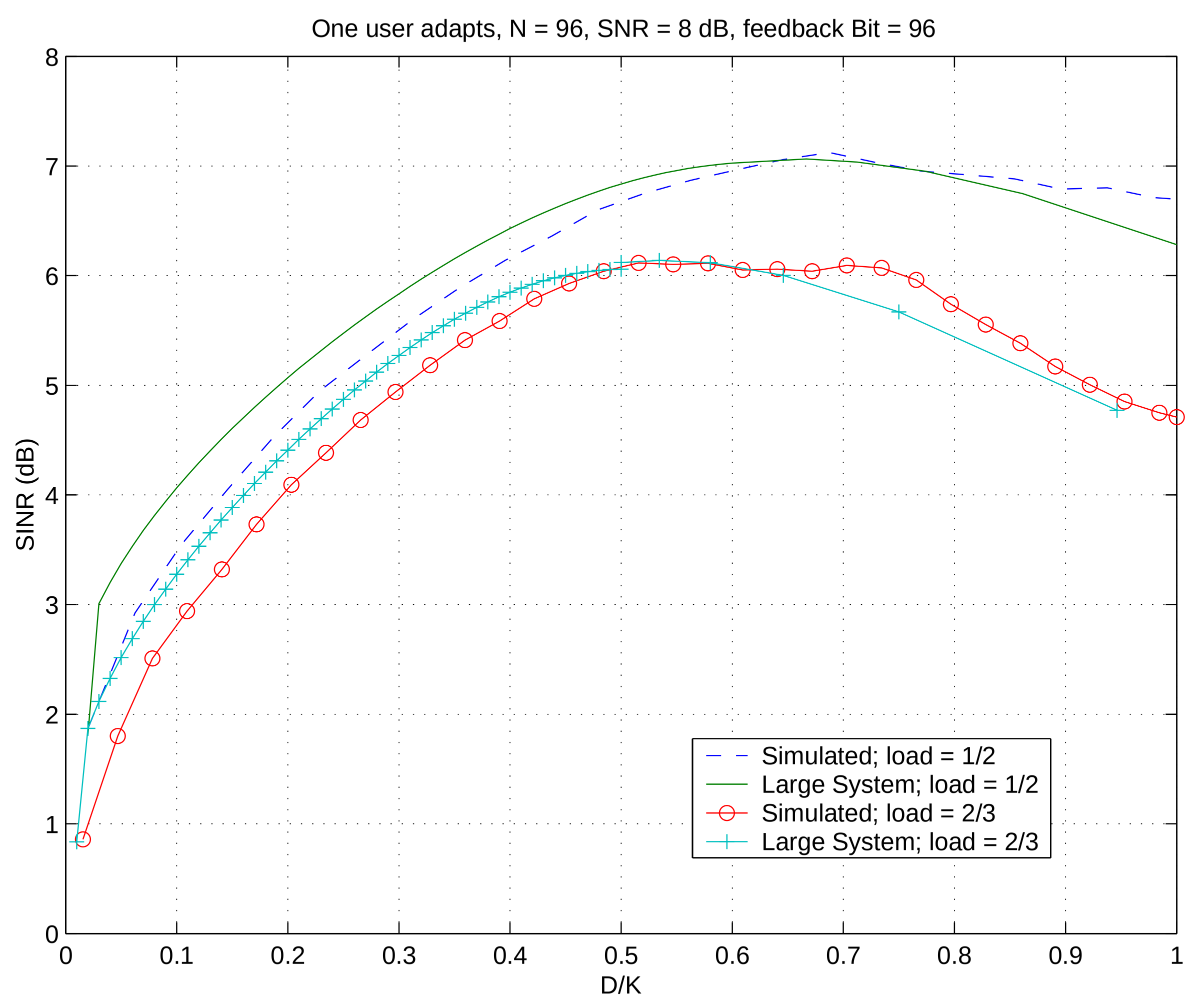
Let $(D, K, N, B) \to \infty$ with fixed

$$\begin{aligned} \bar{K} &= K/N \\ \bar{D} &= D/K \\ \bar{B} &= B/N \end{aligned}$$

$$\begin{aligned} \lim_{K\to\infty} \bar{\beta}_k &= \bar{\beta}_k^\infty \quad \text{(Large System SINR)} \\ &= \frac{1 - E\left[\|\Delta\bar{\boldsymbol{\alpha}}_k\|^2\right]}{\sigma_n^2 + \left(1 - \sqrt{\bar{D}}\right)^2 \bar{K} + \left(2\sqrt{\bar{D}}\bar{K} - \bar{D}\bar{K} - \sigma_n^2\right) E\left[\|\Delta\bar{\boldsymbol{\alpha}}_k\|^2\right]} \end{aligned}$$

Assumes $\Delta\bar{\boldsymbol{\alpha}}_k$ and $\mathbf{S}_k^\dagger \mathbf{F}_k$ are independent.

SINR vs Normalized Dimension: Single-User Optimization



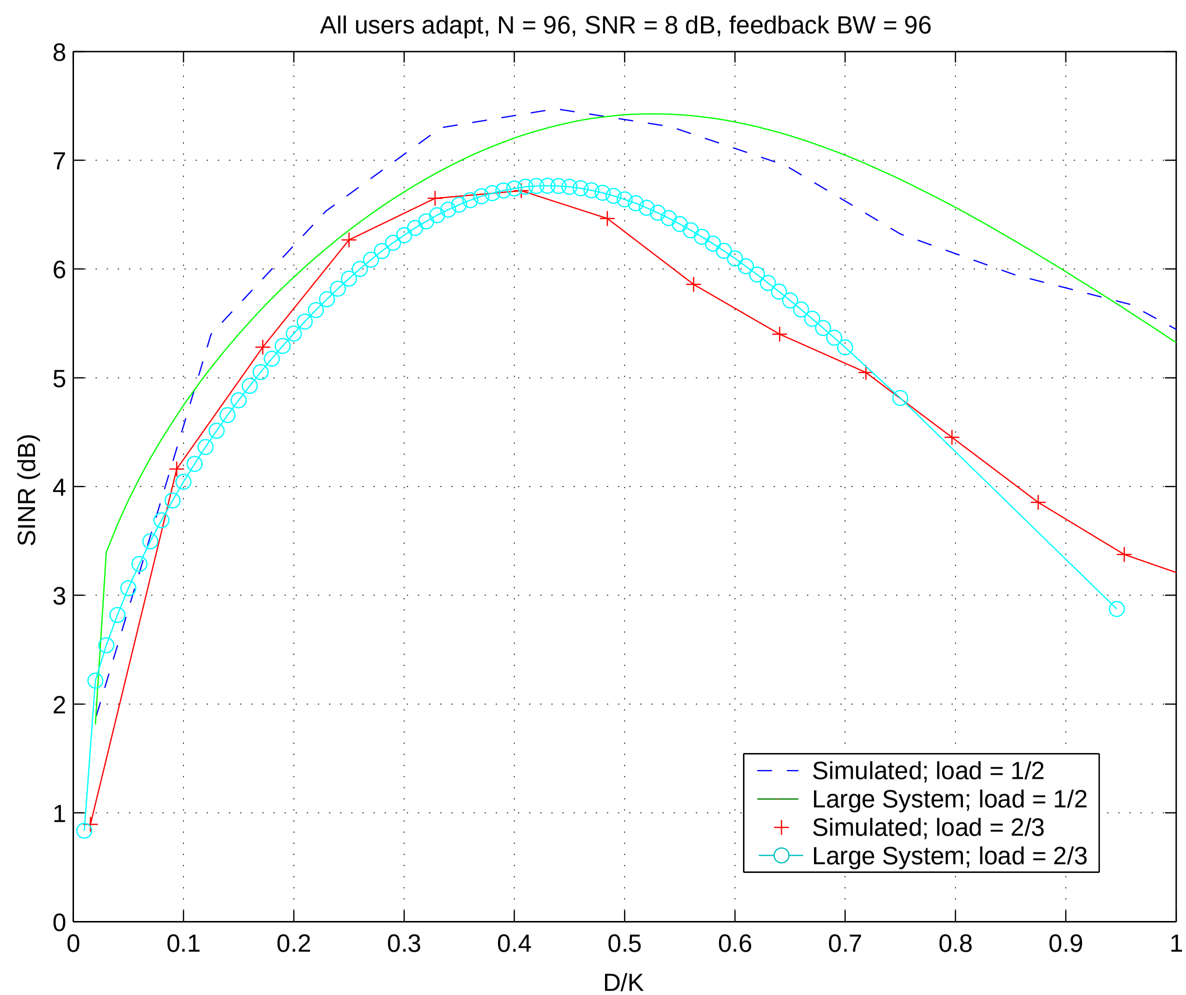
NORTHWESTERN UNIVERSITY

Group Optimization

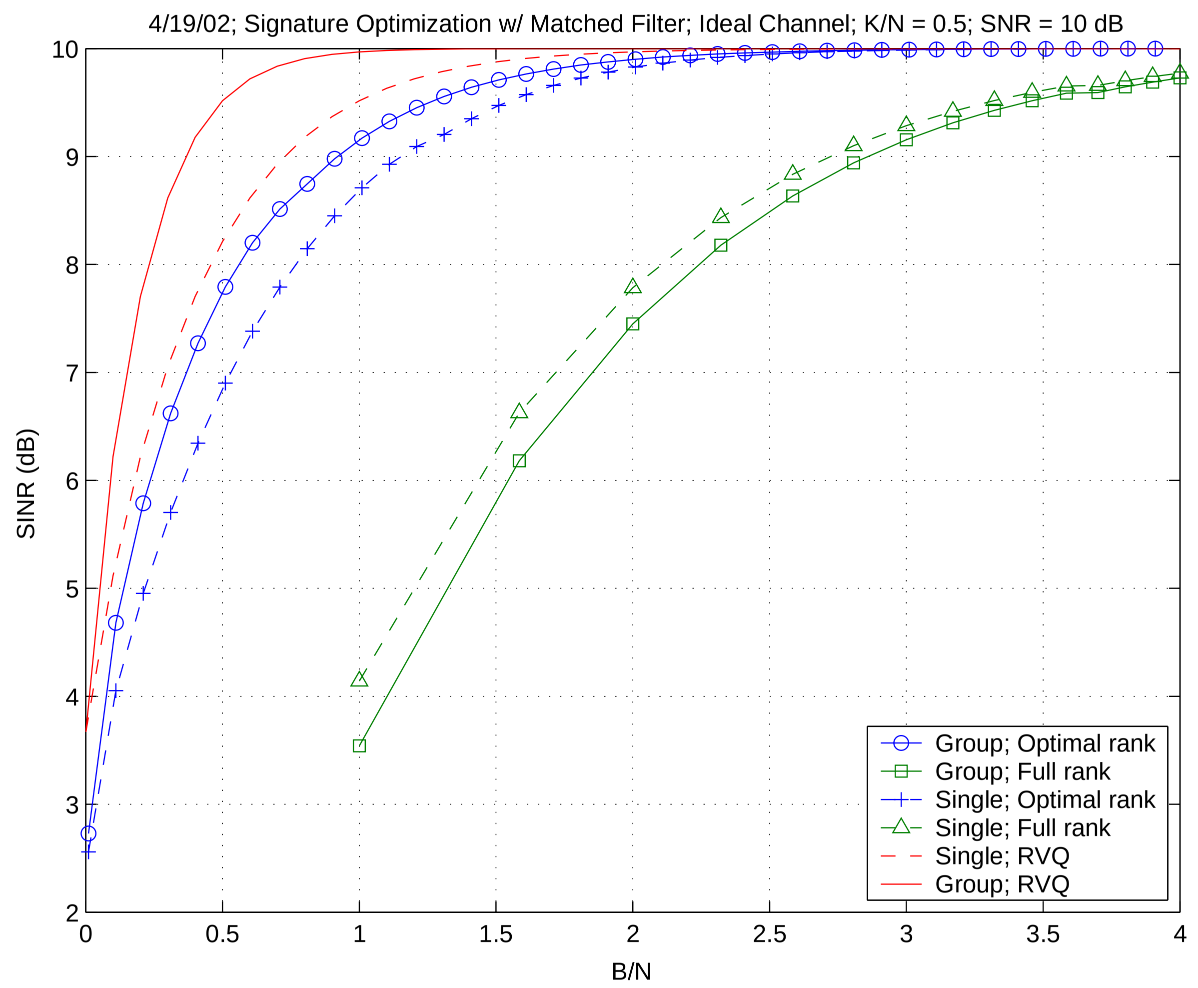
- Each user optimizes signature.
- Approximate large system analysis:
  - Add users sequentially, fix existing signatures.

$$\bar{\beta}^{\infty} \approx \frac{1 - E\left[\|\Delta\bar{\boldsymbol{\alpha}}_k\|^2\right]}{\sigma_n^2 + \bar{K} + \left(-\frac{1}{3}\bar{D}^2\bar{K} + 2\bar{D}\bar{K} - \frac{8}{3}\sqrt{\bar{D}}\bar{K}\right)\left(1 - E\left[\|\Delta\bar{\boldsymbol{\alpha}}_k\|^2\right]\right)}$$

SINR vs Normalized Dimension: Group Optimization



SINR vs B/N: Group Optimization



NORTHWESTERN UNIVERSITY

Adaptive Least Squares Estimation

- System model:

$$\mathbf{r}(i) = \mathbf{S}\sqrt{\mathbf{P}}\mathbf{b}(i) + \mathbf{n}(i)$$

  time index $i = 1, 2, \cdots$

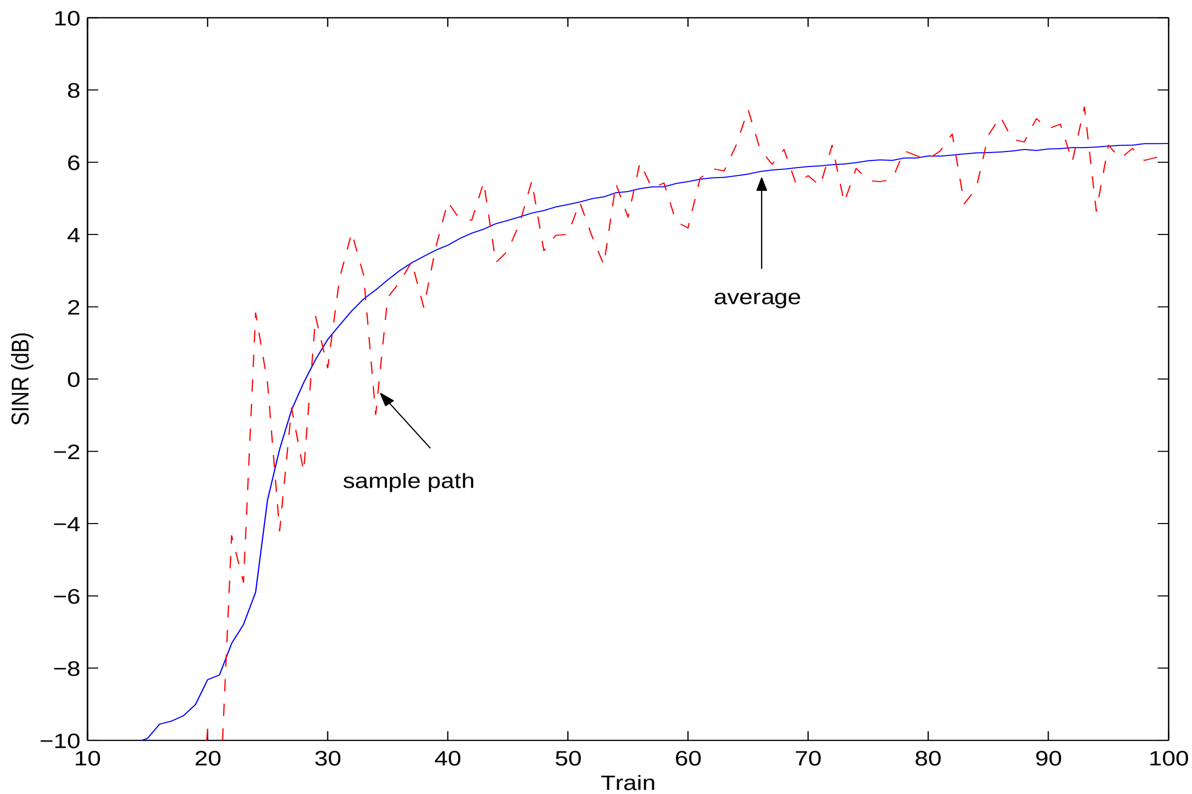
- Linear MMSE receiver: $\mathbf{c}_k = \mathbf{R}^{-1}\mathbf{s}_k$
  - Problem: may not know $\mathbf{R} = \mathbf{SPS}^\dagger + \sigma_n^2\mathbf{I}$
- Least Squares estimates
  - With training:

$$\hat{\mathbf{c}}_k(i) = \arg\min_{\mathbf{c}_k} \sum_{j=1}^{i} |b_k(j) - \mathbf{c}_k^\dagger \mathbf{r}(j)|^2 = \hat{\mathbf{R}}^{-1}(i)\hat{\mathbf{s}}_1(i)$$

    where $\hat{\mathbf{R}}(i) = \sum_{j=1}^{i} \mathbf{r}(j)\mathbf{r}^\dagger(j)$ and $\hat{\mathbf{s}}_k(i) = \sum_{j=1}^{i} b_k^*(j)\mathbf{r}(j)$
  - Without training (blind):

$$\hat{\mathbf{c}}_k(i) = \hat{\mathbf{R}}^{-1}(i)\mathbf{s}_1$$

Convergence: Numerical Example



As $i \to \infty$, $\hat{\mathbf{c}}_k \to \mathbf{c}_k = \mathbf{R}^{-1}\mathbf{s}_k$ $\quad K = 12, N = 24$

SINR $\beta_k(i) = \dfrac{|\hat{\mathbf{c}}_k^\dagger(i)\mathbf{s}_1|^2}{\hat{\mathbf{c}}_k^\dagger(i)\mathbf{R}_k(i)\hat{\mathbf{c}}_k(i)} \to \mathbf{s}_k^\dagger \mathbf{R}_k^{-1}\mathbf{s}_k$

Convergence Problem

- For random $\mathbf{S}$, $\mathbf{b}$, and $\mathbf{n}$, compute $E[\beta_k(i)]$
- Classical problem; very difficult
- Prior work:
  - Reed, Mallet, Brennan (1974): For Gaussian $\mathbf{S}$, $\hat{\beta}_k(i)/\beta_k \approx 1/2$ for $i = 2N$.
  - Asymptotic convergence (large i)
  - Approximate analyses

Large System Convergence Analysis

Let $(K, N, i) \to \infty$ with fixed

$$\begin{aligned} \bar{K} &= K/N \\ \bar{i} &= i/N \end{aligned}$$

- If $\mathbf{S}$ has *i.i.d.* elements, then the SINR converges to an asymptotic limit $\hat{\beta}^{\infty}(\bar{K}, \bar{i})$.
- $\hat{\beta}^{\infty}(\bar{K}, \bar{i}) \to \beta^{\infty}(\bar{K})$ as $\bar{i} \to \infty$.
- Can compute $\hat{\beta}^{\infty}(\bar{K}, \bar{i})$; *independent* of particular distribution of $\mathbf{S}$.
- Accurate for moderate values of N (e.g., ≥ 32).

Output SINR

- With training:

$$\hat{\beta}^{\infty}(\bar{K},\bar{i}) = \frac{\beta^{\infty}(\bar{K})}{1+\frac{1}{\bar{i}-1}\left(1+\frac{1}{\beta^{\infty}(\bar{K})}\right)}$$

- Without training:

$$\hat{\beta}^{\infty}(\bar{K},\bar{i}) = \frac{\beta^{\infty}(\bar{K})}{1+\frac{1}{\bar{i}-1}\left(1+\beta^{\infty}(\bar{K})\right)}$$

- $\bar{i} = i/N = 1 \implies \hat{\beta}^{\infty} = 0$
- With training and large β^{∞}, $\hat{\beta}^{\infty} \approx \beta^{\infty}(\bar{i}-1)/\bar{i}$
  $\bar{i} = 2 \implies \hat{\beta}^{\infty} \approx \beta^{\infty}/2$
- Without training, SINR is *estimation-error limited* (Zhang & Wang)
- When $\beta^{\infty} > 1$, training is best, and vice versa when $\beta^{\infty} < 1$.

Derivation Outline

- Problem reduces to determining the asymptotic eigenvalue distribution of $\hat{\mathbf{R}}(i) = \frac{1}{i}\sum_{j=1}^{i}\mathbf{r}(j)\mathbf{r}^{\dagger}(j)$.
- Without noise,

$$\begin{aligned}\hat{\mathbf{R}}(i) &= \frac{1}{i}\mathbf{S}\mathbf{B}(i)\mathbf{P}\mathbf{B}^{\dagger}(i)\mathbf{S}^{\dagger} \\ &= \mathbf{S}\mathbf{V}_B(i)\Lambda_B(i)\mathbf{V}_B^{\dagger}(i)\mathbf{S}^{\dagger} \\ &= \mathbf{V}_S(i)\Lambda_B(i)\mathbf{V}_S^{\dagger}\end{aligned}$$

  where $\mathbf{B}(i) = [\mathbf{b}(1)\cdots\mathbf{b}(i)]$.
- For Gaussian $\mathbf{S}$, $\mathbf{V}_S(i)$ is *i.i.d*, and the diagonal elements of $\Lambda_B(i)$ converge to the eigenvalue distribution of $\mathbf{B}\mathbf{P}\mathbf{B}^{\dagger}$.
- "Recursive" application of Silverstein theorem.
- Can generalize to include additive noise.

Asymptotic Eigenvalue Distribution of $\hat{\mathrm{R}}$

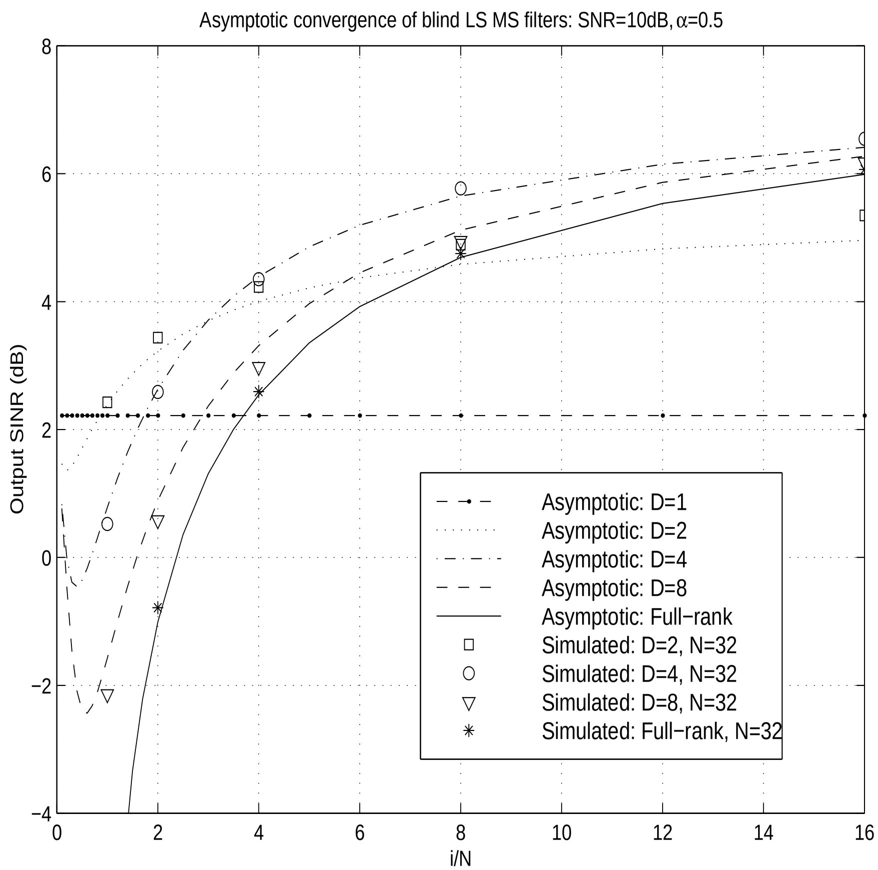
- Stieltjes transform satisfies

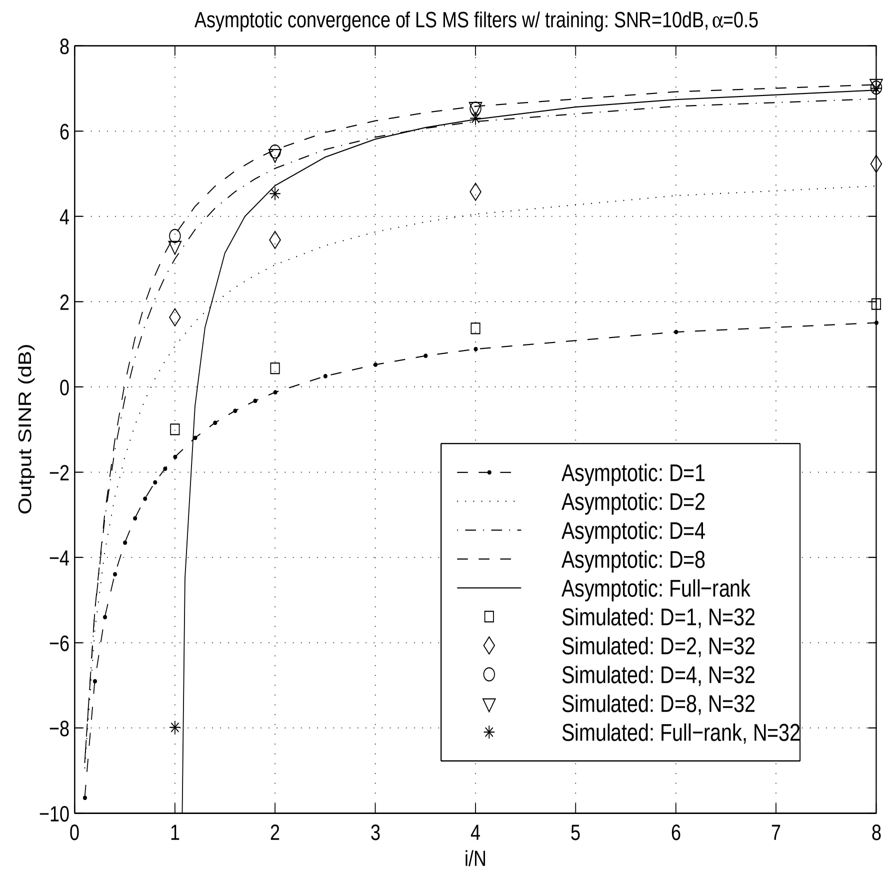
$$m_{\hat{\mathbf{R}}}(z)\left(\frac{\bar{i}-1-m_{\hat{\mathbf{R}}}(z)z}{\bar{i}}\right)=\frac{1+m_{\hat{\mathbf{R}}}(z)z}{\sigma^2+\bar{K}\int\frac{P\,dF_P(P)}{1+Pm_{\hat{\mathbf{R}}}(z)\left(\frac{\bar{i}-1-m_{\hat{\mathbf{R}}}(z)z}{\bar{i}}\right)}}$$

- Efficient combinatorial method for computing moments

$\lim_{N\to\infty}\mathrm{trace}\{\frac{1}{N}\hat{\mathbf{R}}^n(i)\}$

Large System Convergence Plots





$\bar{K} = 1/2$, SNR=10 dB

Extensions

- Data windowing:

$$\hat{\mathbf{c}}_k(i) = \arg\min_{\mathbf{c}_k} \sum_{j=1}^{i} w_j |b_k(i) - \mathbf{c}_k^\dagger \mathbf{r}(i)|^2$$

  where the data windowing sequence $\{w_1, \cdots, w_i\} \xrightarrow{\mathcal{D}} F_w(\cdot)$ as $i \to \infty$

  – Performance evaluation reduces to determining the asymptotic eigenvalue distribution, or moments, of $\mathbf{WVPV}$, where $\mathbf{V}$ is *i.i.d.*, and $\mathbf{W}$ and $\mathbf{P}$ are diagonal.

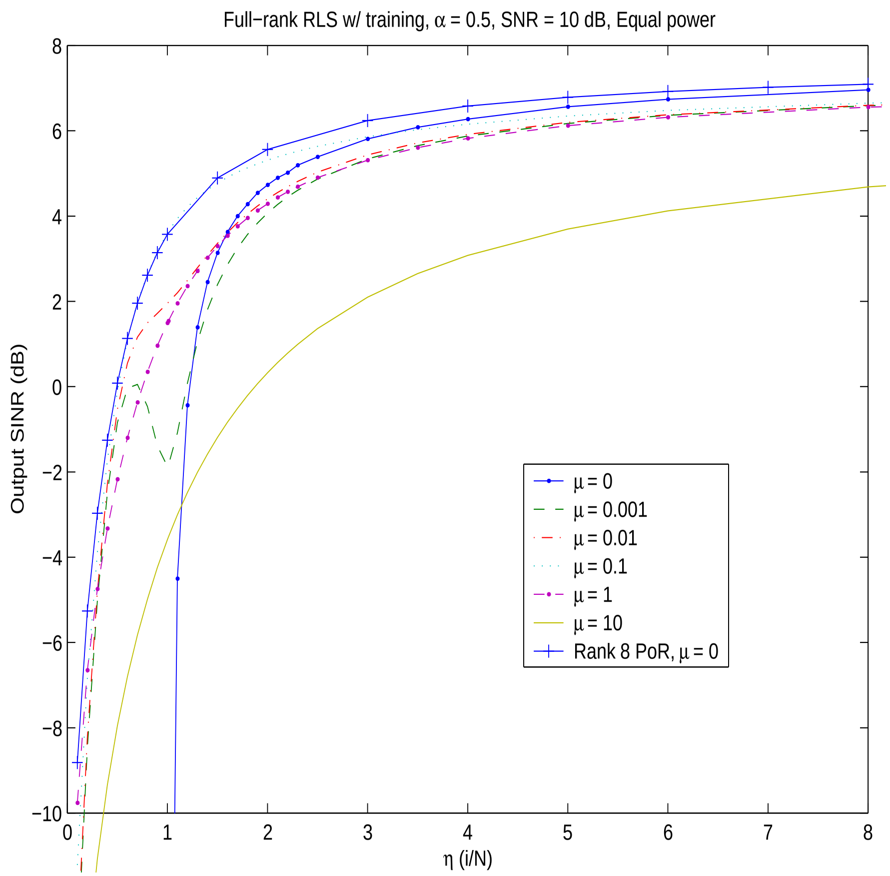
- Diagonal loading: $\hat{\mathbf{R}}(i) = \sum_{j=1}^{i} \mathbf{r}(j)\mathbf{r}^\dagger(j) + \delta\mathbf{I}$ where $\delta > 0$ is the diagonal loading factor, and prevents ill-conditioning.

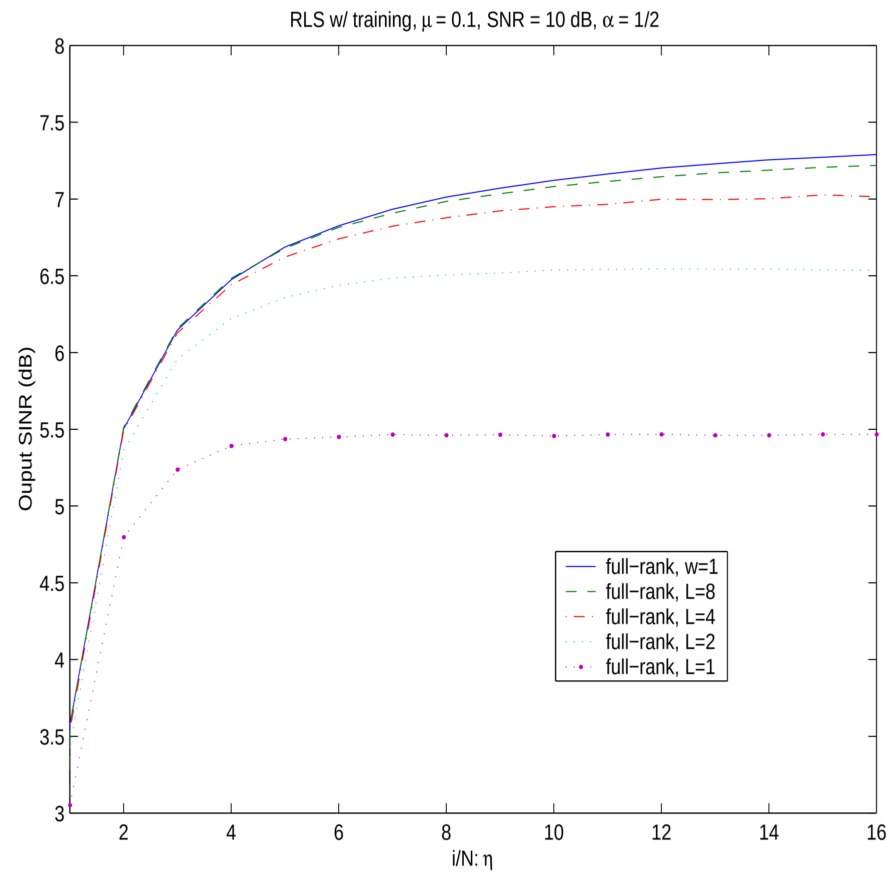
- Reduced-rank least squares filtering

  – Constrain filter to lie in lower-dimensional Krylov subspace spanned by $[\hat{\mathbf{s}}_1\ \hat{\mathbf{R}}\hat{\mathbf{s}}_1 \cdots \hat{\mathbf{R}}^{D-1}\hat{\mathbf{s}}_1]$.

  – Can reduce complexity, improve convergence.

Effect of Diagonal Loading and Windowing





$\bar{K} = 1/2$, SNR=10 dB, with training

Asymptotic Limits: Recap

- CDMA ($\mathbf{r} = \mathbf{S}\mathbf{b} + \mathbf{n}$):
  - Users, Degrees of Freedom (DoF) $\rightarrow \infty$
- Multi-Carrier-CDMA ($\mathbf{r} = \sum_{j=1}^{J} \mathbf{H}_j \mathbf{S}_j \mathbf{b}_j + \mathbf{n}$):
  - Signatures per user, DoF $\rightarrow \infty$ (fix users)
- Signature optimization with limited feedback
  - Users, feedback bits, DoF $\rightarrow \infty$
  - Reduced-rank dimension, users, feedback bits, DoF $\rightarrow \infty$
- Convergence analysis of adaptive Least Squares estimator
  - Training interval, users, DoF $\rightarrow \infty$

Conclusions

- Can sometimes characterize the performance of finite-size systems in terms of large system limits.
- Insights:
  - Sum spectral efficiency of multi-user MC-CDMA decreases with signatures per user.
  - Performance can be substantially improved with one bit of feedback per DoF.
  - Least Squares estimation with training performs better than without training when the optimal SINR > 1 (and vice versa).
- Asymptotic analysis using random matrix theory is an active area.