Diversity and Multiplexing:
A Fundamental Tradeoff in Wireless Systems

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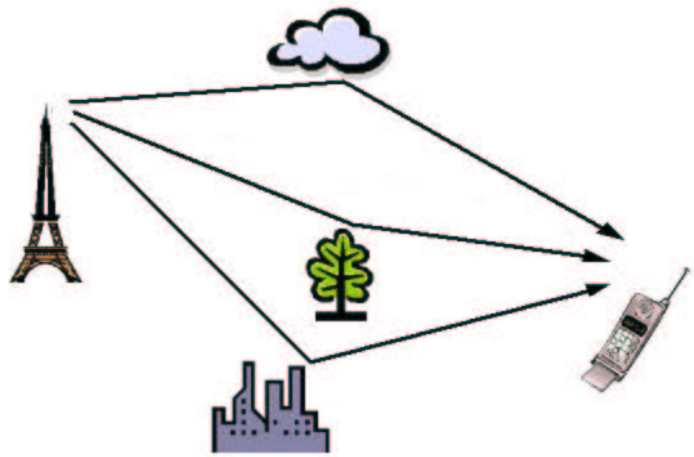
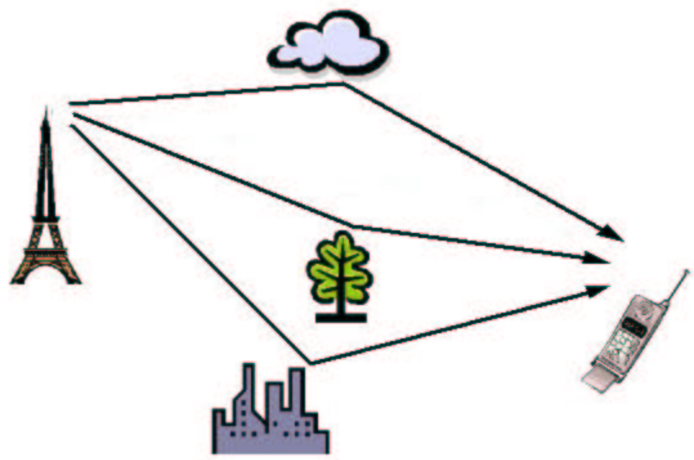
Wireless Fading Channels



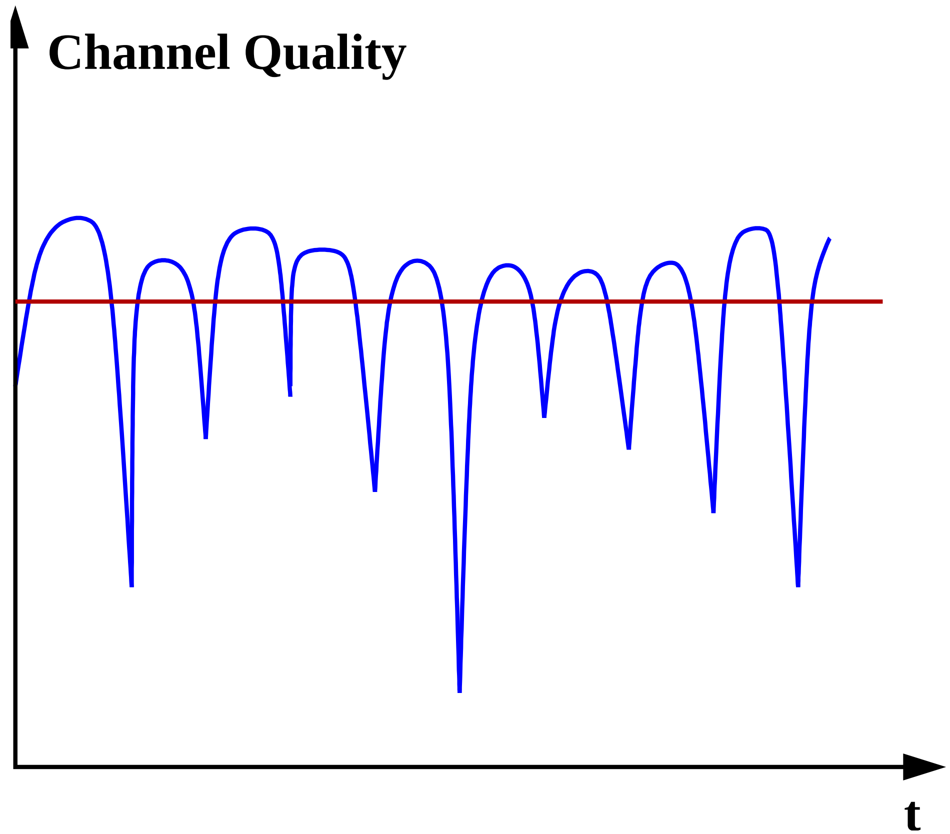
- Fundamental characteristic of wireless channels: multi-path fading.

Wireless Fading Channels



- Fundamental characteristic of wireless channels: multi-path fading.
- Two important resources of a fading channel: diversity and degrees of freedom.

Diversity



A channel with more diversity has smaller probability in deep fades.

Example: Spatial Diversity

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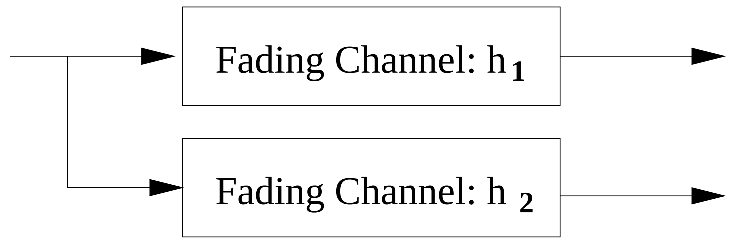
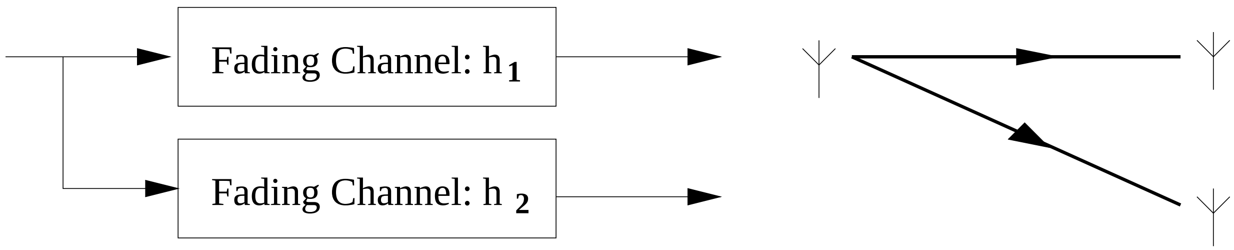


- Additional independent fading channels increase diversity.

Example: Spatial Diversity



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- Spatial diversity

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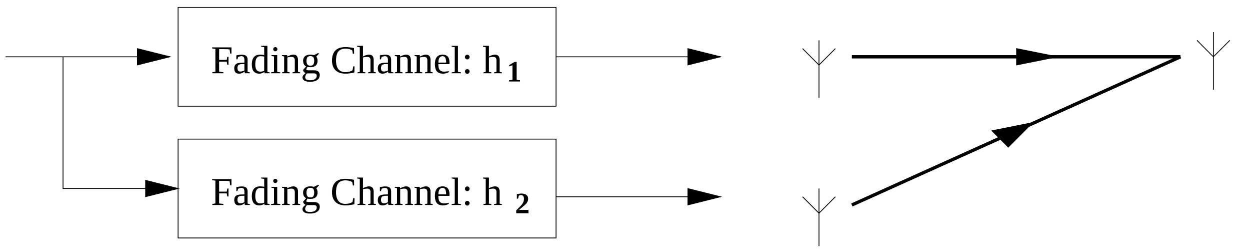

- Additional independent fading channels increase diversity.
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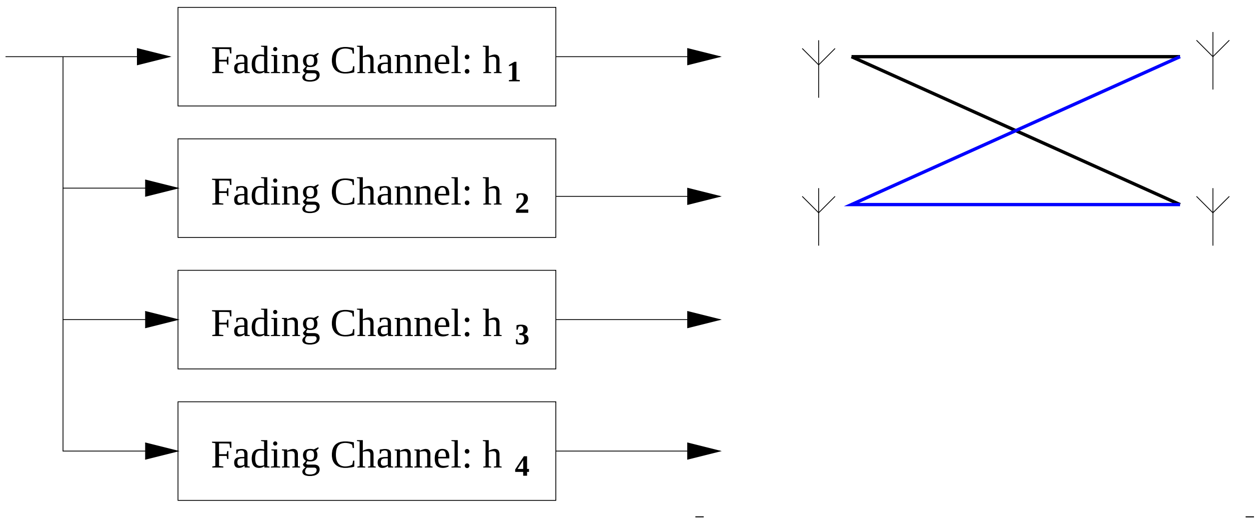

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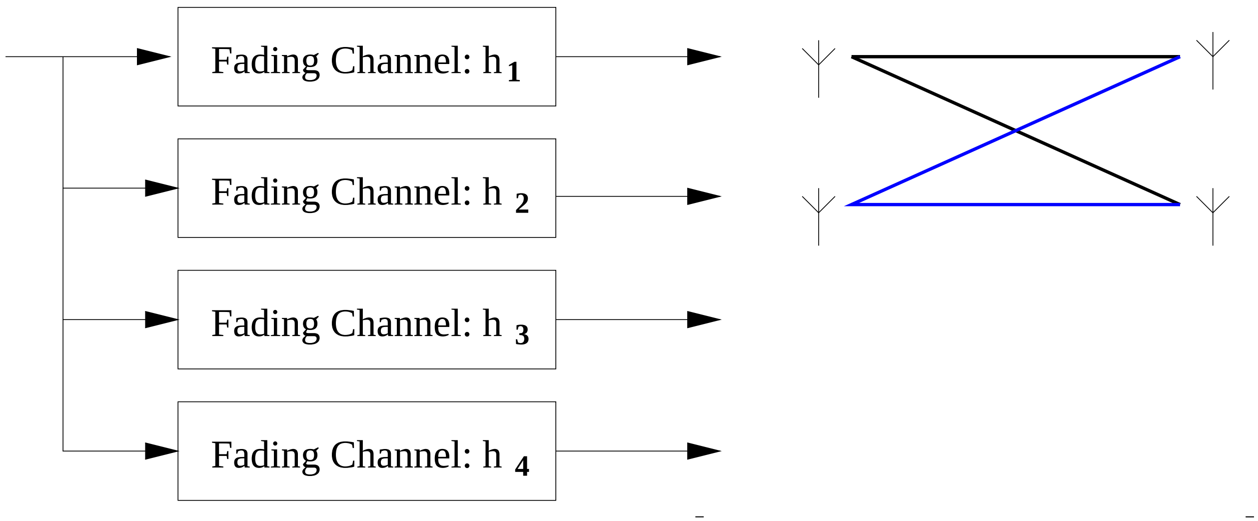
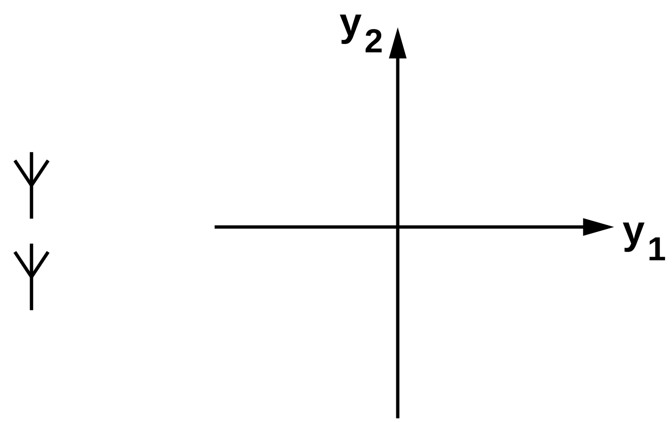


- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- Repeat and Average: compensate against channel unreliability.

Degrees of Freedom



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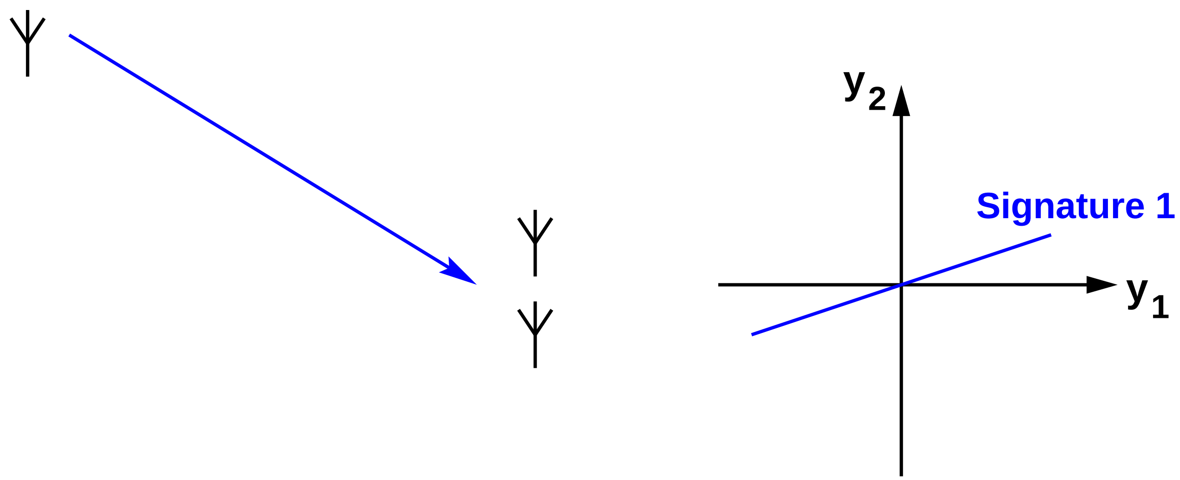

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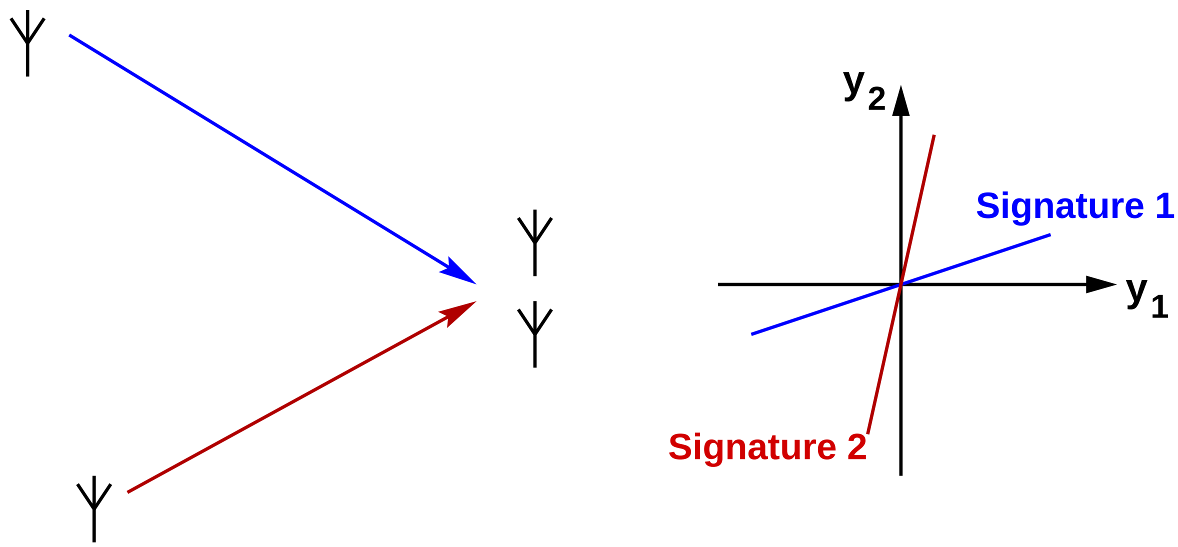
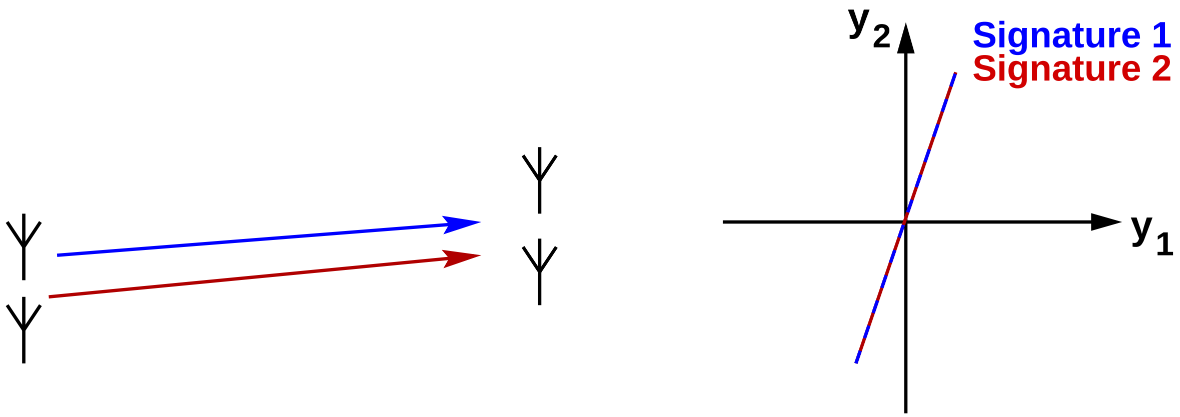


Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Degrees of Freedom



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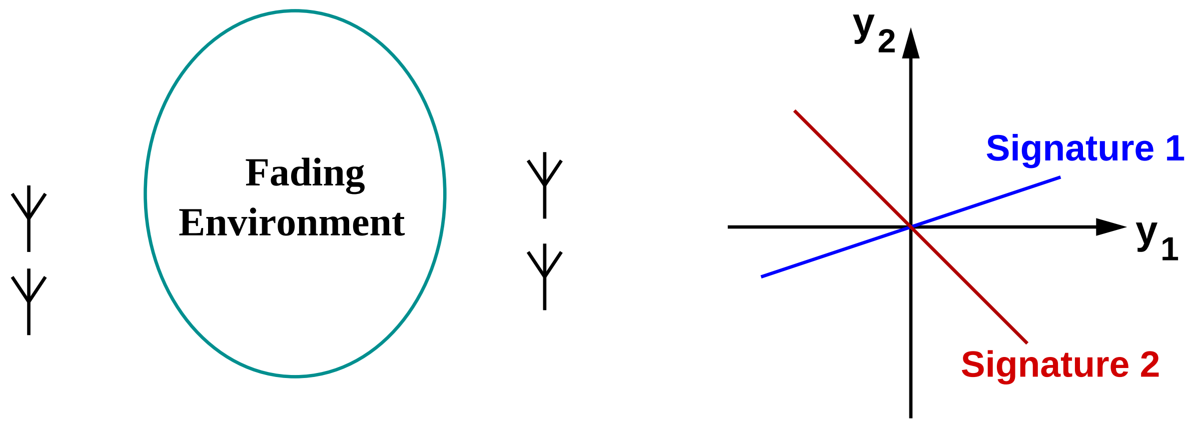
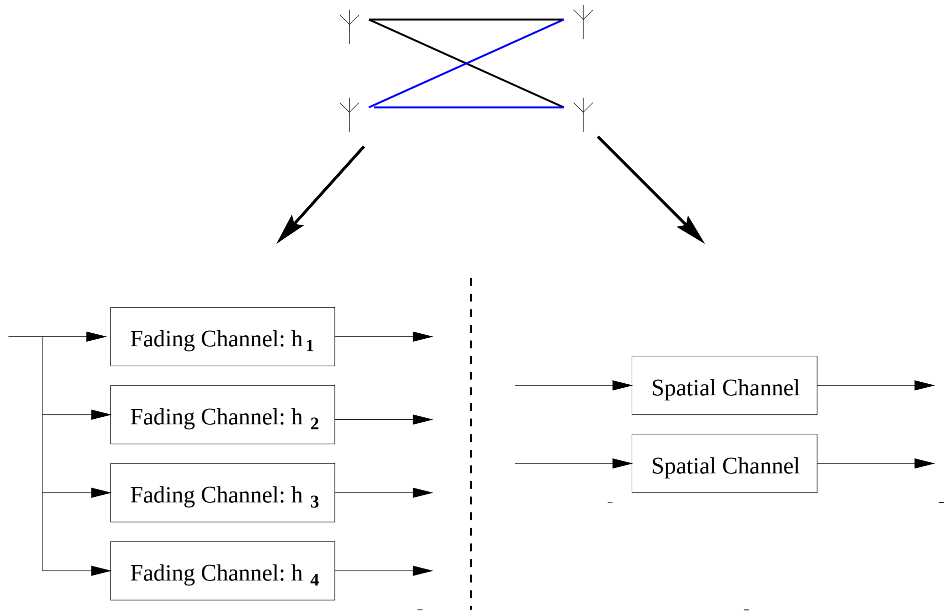


Signals arrive in multiple directions provide multiple degrees of freedom for communication.

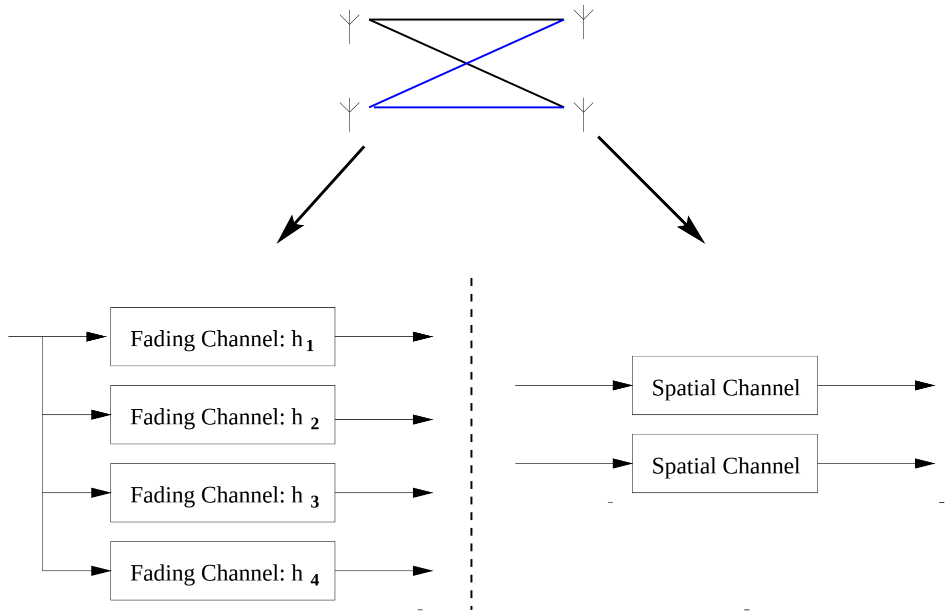
Same effect can be obtained via scattering even when antennas are close together.

Diversity vs. Multiplexing



The two resources have been considered mainly in isolation: existing schemes focus on maximizing either the diversity gain or the multiplexing gain.

Diversity vs. Multiplexing



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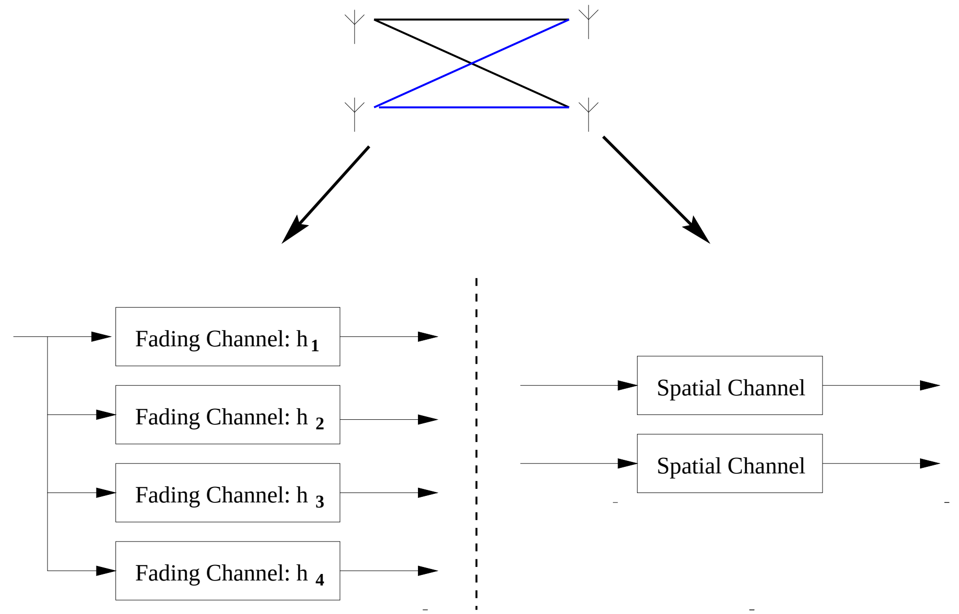

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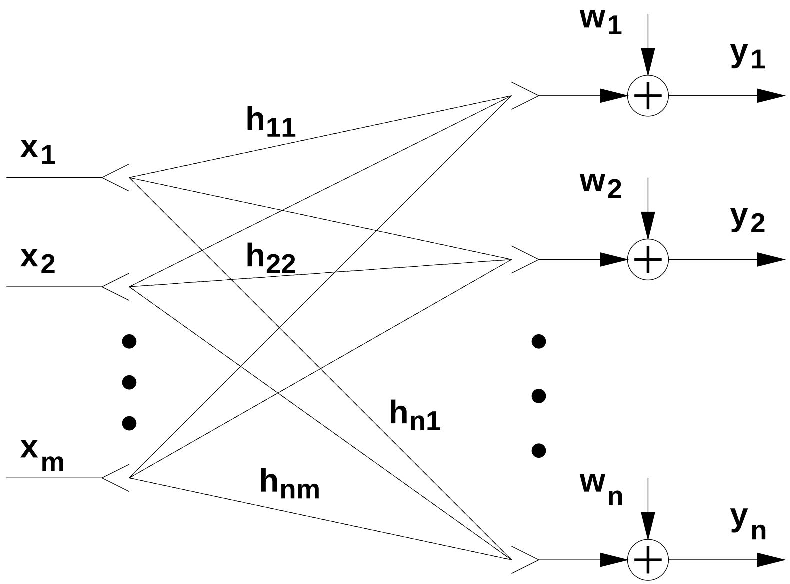
The right way of looking at the problem is a tradeoff between the two types of gain.

The optimal tradeoff achievable by a coding scheme gives a fundamental performance limit on communication over fading channels.

Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- cooperative relaying systems

Point-to-point MIMO Channel



$$\mathbf{y}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{w}_t, \qquad \mathbf{w}_t \sim \mathcal{CN}(0,1)$$

- Rayleigh flat fading i.i.d. across antenna pairs ($h_{ij} \sim \mathcal{CN}(0,1)$).
- SNR is the average signal-to-noise ratio at each receive antenna.

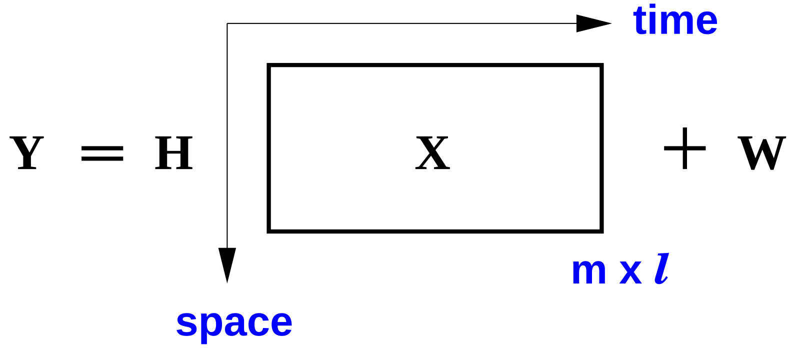
Coherent Block Fading Model

- Focus on codes over l symbols, where $\mathbf{H}$ remains constant.
- $\mathbf{H}$ is known to the receiver but not the transmitter.
- Assumption valid as long as

$$l \ll \text{coherence time} \times \text{coherence bandwidth}.$$

Space-Time Block Code

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$$



Focus on coding over a single block of length l.

Diversity Gain

Motivation: Binary Detection

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w} \qquad\qquad P_e \approx P(\|\mathbf{h}\| \text{ is small }) \propto \mathsf{SNR}^{-1}$$

$$\left.\begin{aligned} \mathbf{y}_1 &= \mathbf{h}_1\mathbf{x} + \mathbf{w}_1 \\ \mathbf{y}_2 &= \mathbf{h}_2\mathbf{x} + \mathbf{w}_2 \end{aligned}\right\} \qquad \begin{aligned} P_e \quad &\approx P(\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \text{ are both small}) \\ &\propto \mathsf{SNR}^{-2} \end{aligned}$$

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General Definition

A space-time coding scheme achieves diversity gain d, if

$$P_e(\text{SNR}) \sim \text{SNR}^{-d}$$

Spatial Multiplexing Gain

Motivation: Channel capacity (Telatar '95, Foschini'96)

$$C(\mathsf{SNR}) \approx \min\{m,n\} \log \mathsf{SNR} (bps/Hz)$$

$\min\{m,n\}$ degrees of freedom to communicate.

Spatial Multiplexing Gain

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$$C(\mathsf{SNR}) \approx \min\{m, n\} \log \mathsf{SNR}(bps/Hz)$$

$\min\{m, n\}$ degrees of freedom to communicate.

Definition A space-time coding scheme achieves spatial multiplexing gain r, if

$$R(\mathsf{SNR}) = r \log \mathsf{SNR}(bps/Hz)$$

Fundamental Tradeoff

A space-time coding scheme achieves

Spatial Multiplexing Gain r : $R = r \log \mathsf{SNR} \quad (bps/Hz)$

and

Diversity Gain d : $P_e \approx \mathsf{SNR}^{-d}$

Fundamental Tradeoff

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Fundamental tradeoff: for any r, the maximum diversity gain achievable: $d^*_{m,n}(r)$.

$$r \longrightarrow d^*_{m,n}(r)$$

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$$r \longrightarrow d^*_{m,n}(r)$$

A tradeoff between data rate and error probability.

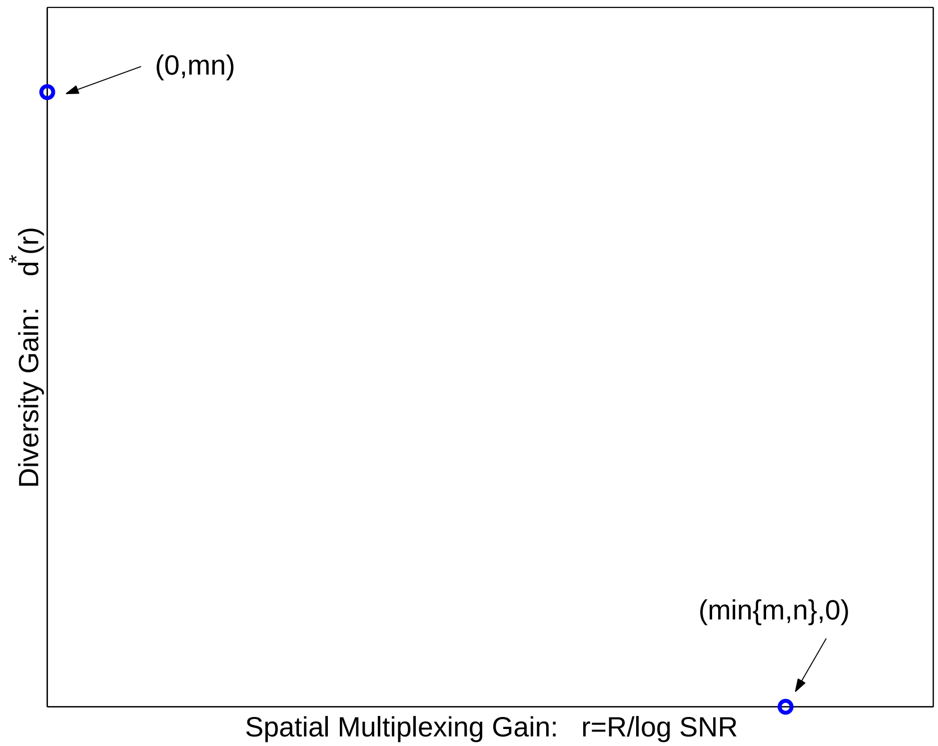
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

m: # of Tx. Ant.
n: # of Rx. Ant.
l: block length
$l \geq m + n - 1$

d: diversity gain
$P_e \approx \mathsf{SNR}^{-d}$

r: multiplexing gain
$R = r \log \mathsf{SNR}$



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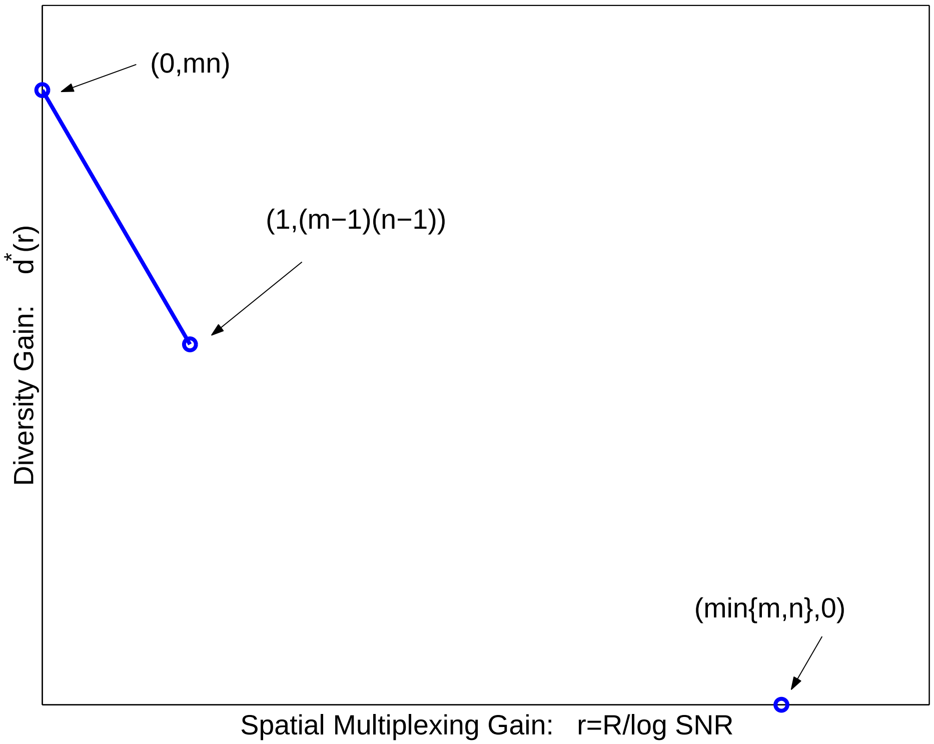

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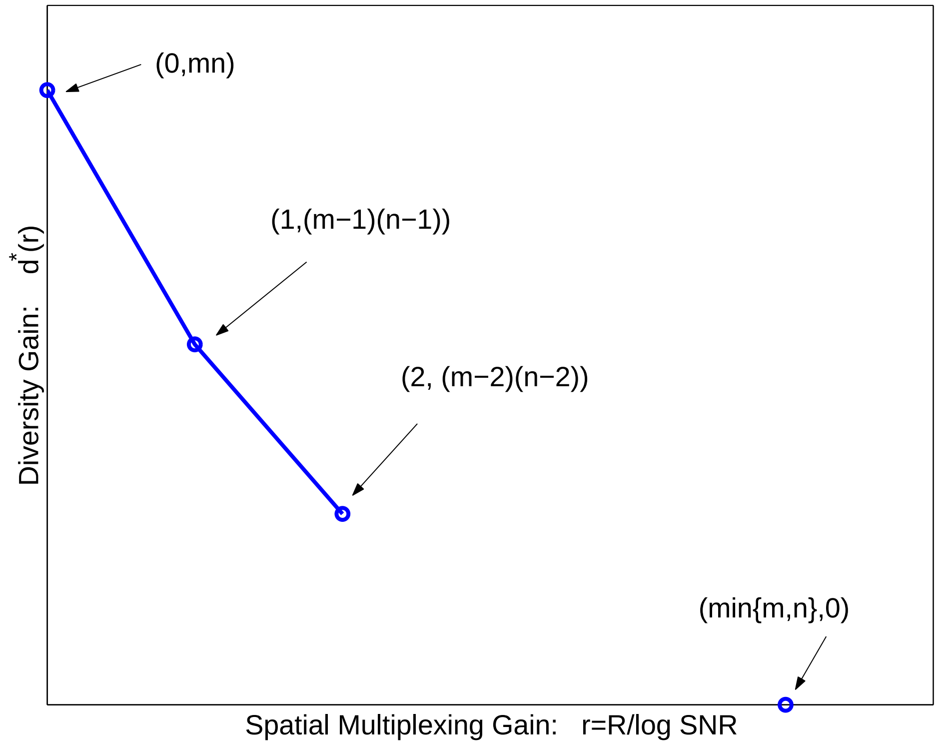

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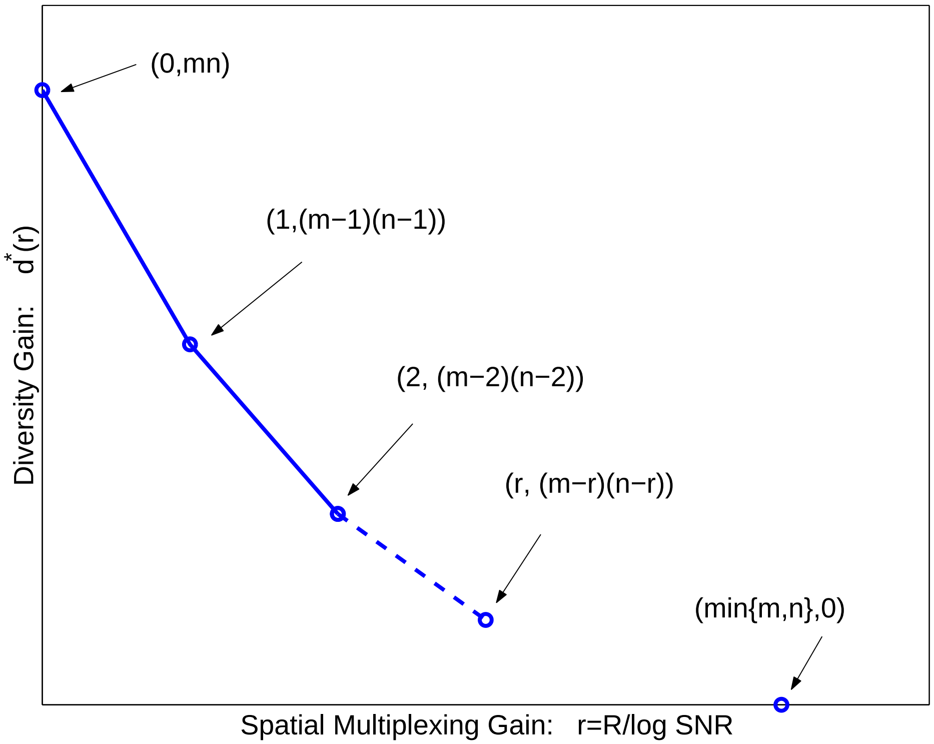

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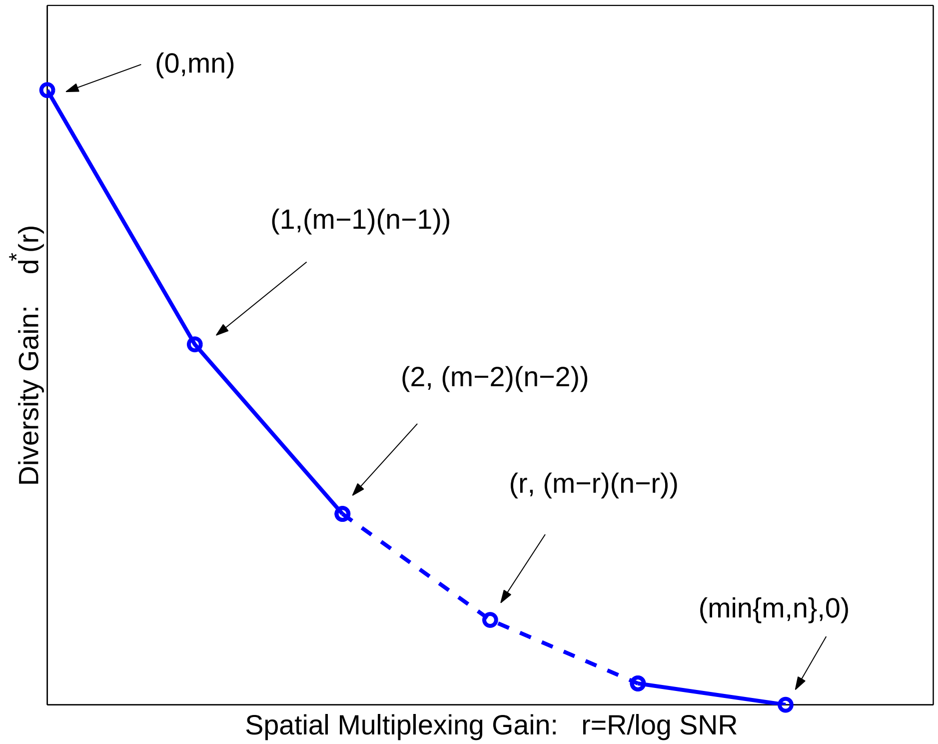

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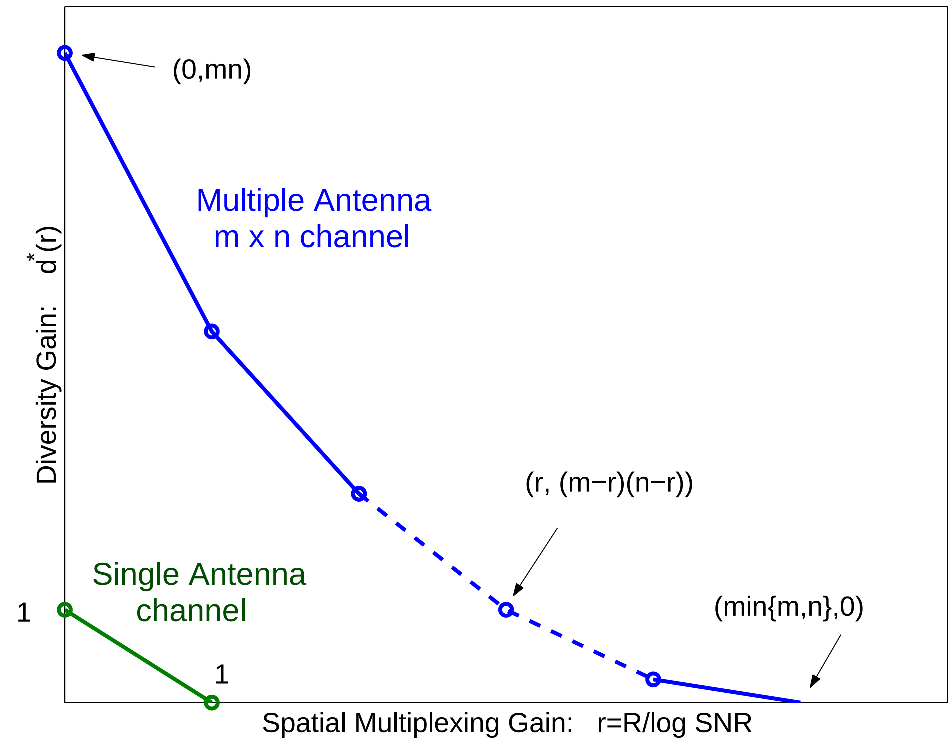

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What do I get by adding one more antenna at the transmitter and the receiver?

Adding More Antennas

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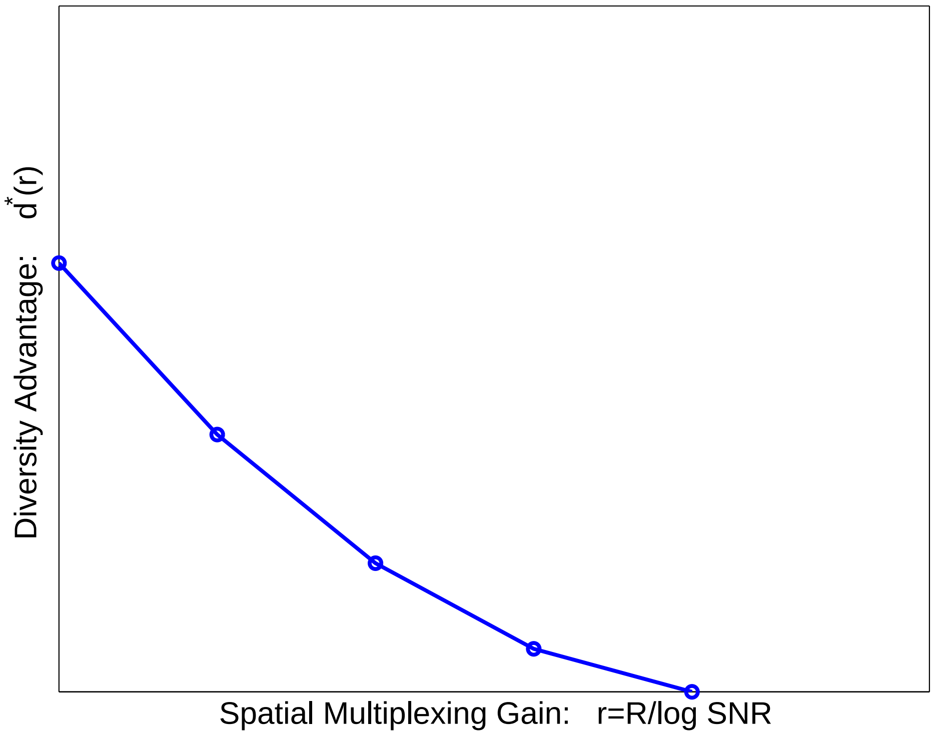

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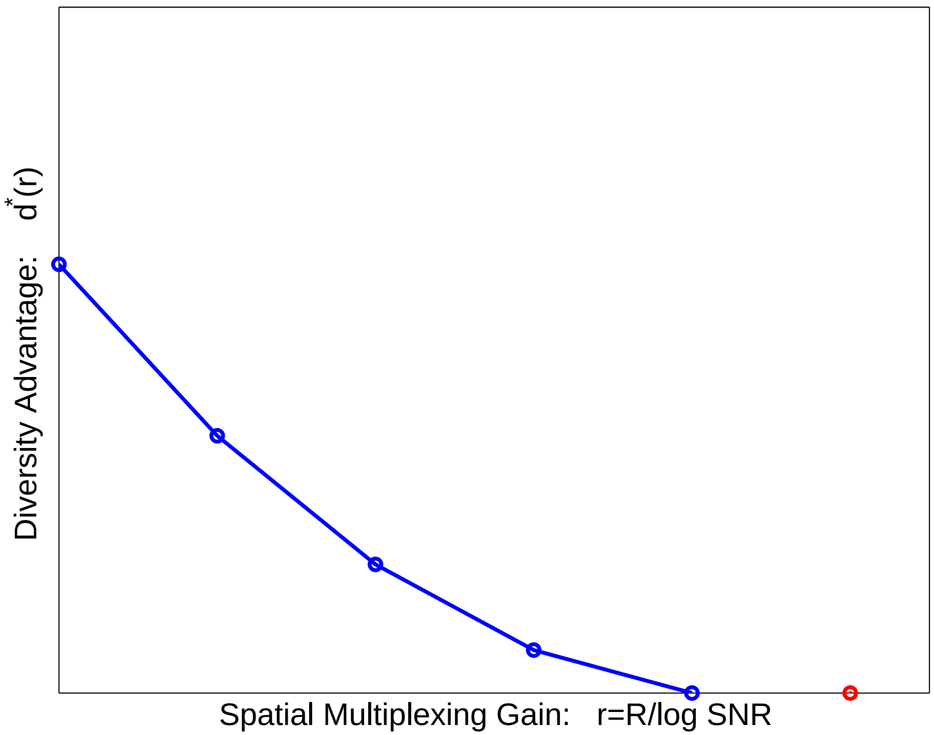

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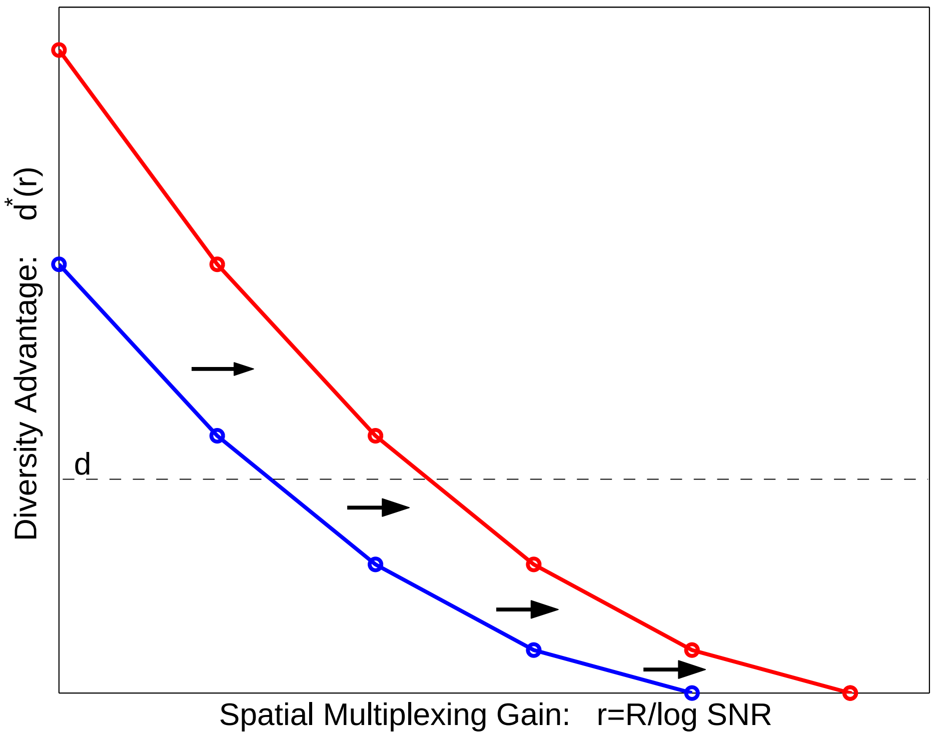

- Capacity result: increasing $\min\{m, n\}$ by 1 adds 1 more degree of freedom.
- Tradeoff curve: increasing both m and n by 1 yields multiplexing gain $+1$ for any diversity requirement d.

Sketch of Proof

Lemma:

For block length $l \geq m + n - 1$, the error probability of the best code satisfies at high SNR:

$$P_e(\mathsf{SNR}) \approx P(\mathsf{Outage}) = P(I(H) < R)$$

where

$$I(H) = \log\det\left[I + \mathsf{SNR}\mathbf{H}\mathbf{H}^*\right]$$

is the mutual information achieved by the i.i.d. Gaussian input.

Outage Analysis

$$P(\text{Outage}) = P\{\log\det[I + \text{SNR}\mathbf{H}\mathbf{H}^\dagger] < R\}$$

- In scalar 1×1 channel, outage occurs when the channel gain $\|\mathbf{h}\|^2$ is small.

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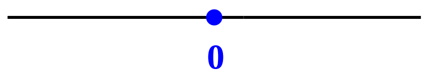
- In general $m \times n$ channel, outage occurs when some or all of the singular values of $\mathbf{H}$ are small. There are many ways for this to happen.
- Let $\mathbf{v} =$ vector of singular values of $\mathbf{H}$:
  Laplace Principle:

$$P(\text{Outage}) \approx \min_{\mathbf{v} \in \mathbf{Out}} \mathsf{SNR}^{-f(\mathbf{v})}$$

Geometric Picture (integer r)

Scalar Channel



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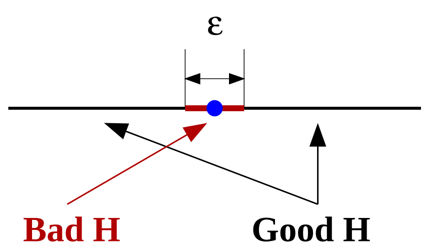

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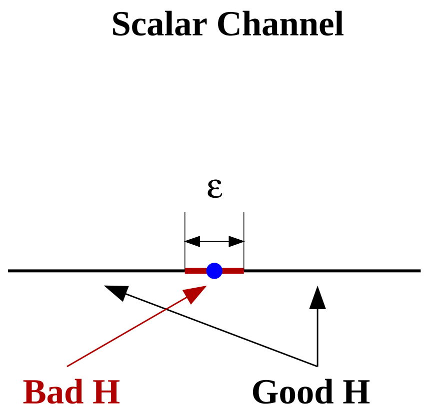

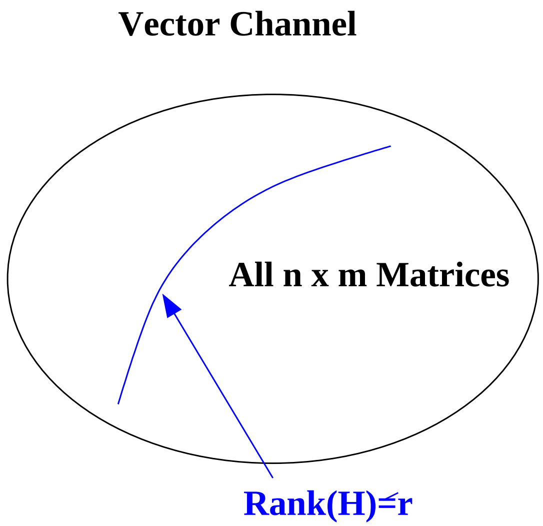

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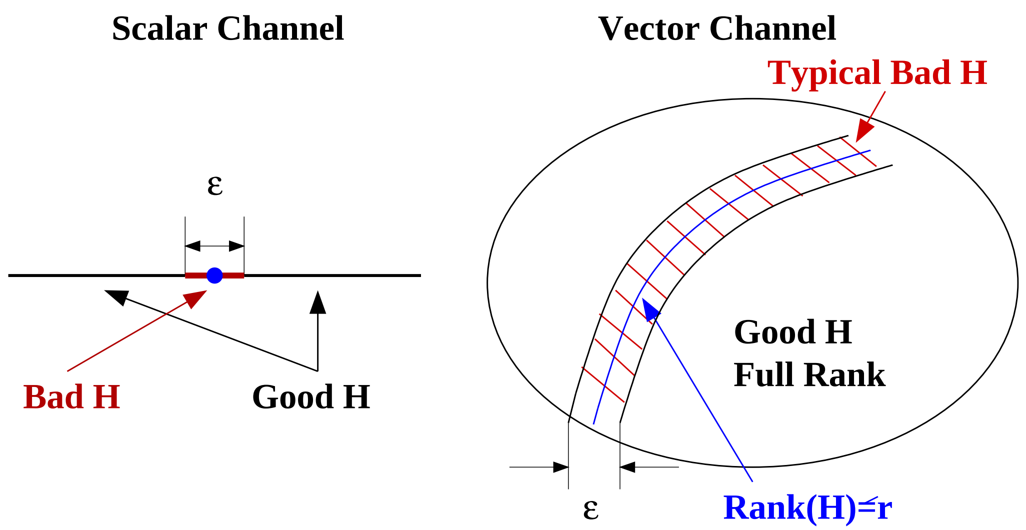

Result: At rate $R = r \log \text{SNR}$, for r integer, outage occurs typically when $\mathbf{H}$ is close to the set $\{\mathbf{H} : rank(\mathbf{H}) \leq r\}$,

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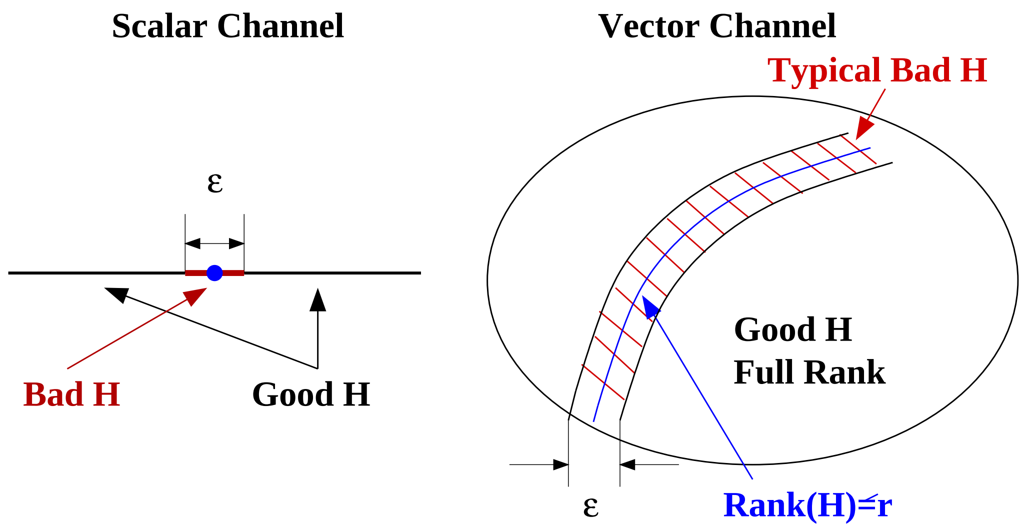

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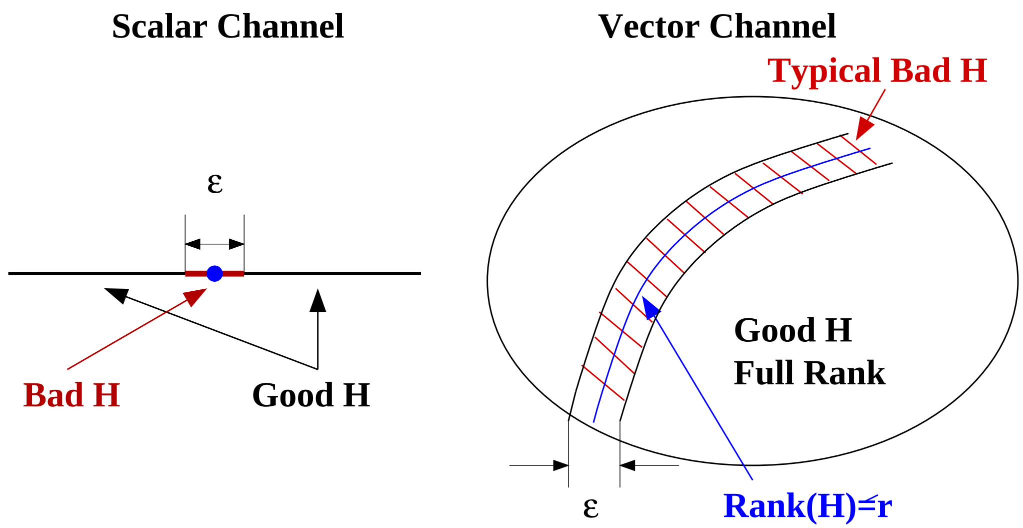

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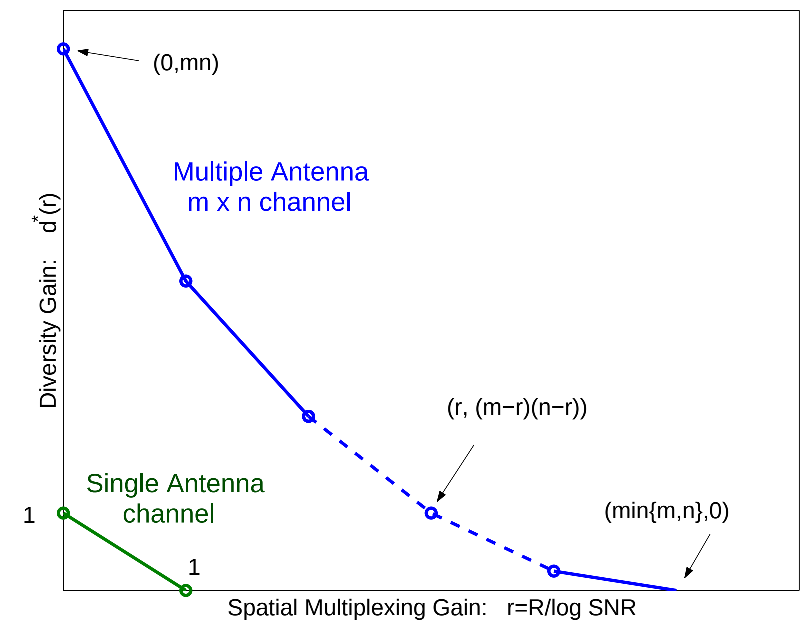
The co-dimension of the manifold of rank r matrices within the set of all $m \times n$ matrices is $(m-r)(n-r)$.

$$P(\text{Outage}) \approx \mathsf{SNR}^{-(m-r)(n-r)}$$

Piecewise Linearity of Tradeoff Curve



For non-integer r, qualitatively same outage behavior as $\lfloor r \rfloor$ but with larger ϵ.

Scalar channel: qualitatively same outage behavior for all r.

Vector channel: qualitatively different outage behavior in different segments of the tradeoff curve.

Tradeoff Analysis of Specific Designs

Focus on two transmit antennas.

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$$

Repetition Scheme:

$$\mathbf{X} = \begin{array}{l} \xrightarrow{\text{time}} \\ \begin{bmatrix} \mathbf{x}_1 & 0 \\ 0 & \mathbf{x}_1 \end{bmatrix} \\ \downarrow \text{space} \end{array}$$

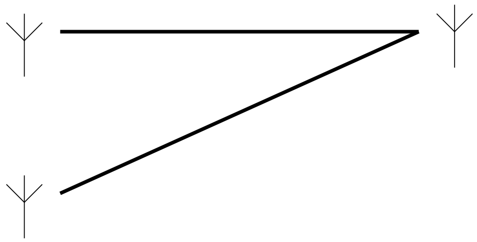
$$\mathbf{y}_1 = \|\mathbf{H}\|\mathbf{x}_1 + \mathbf{w}_1$$

Alamouti Scheme:

$$\mathbf{X} = \begin{array}{l} \xrightarrow{\text{time}} \\ \begin{bmatrix} \mathbf{x}_1 & -\mathbf{x}_2^* \\ \mathbf{x}_2 & \mathbf{x}_1^* \end{bmatrix} \\ \downarrow \text{space} \end{array}$$

$$[\mathbf{y}_1 \mathbf{y}_2] = \|\mathbf{H}\|[\mathbf{x}_1 \mathbf{x}_2] + [\mathbf{w}_1 \mathbf{w}_2]$$

Comparison: 2×1 System



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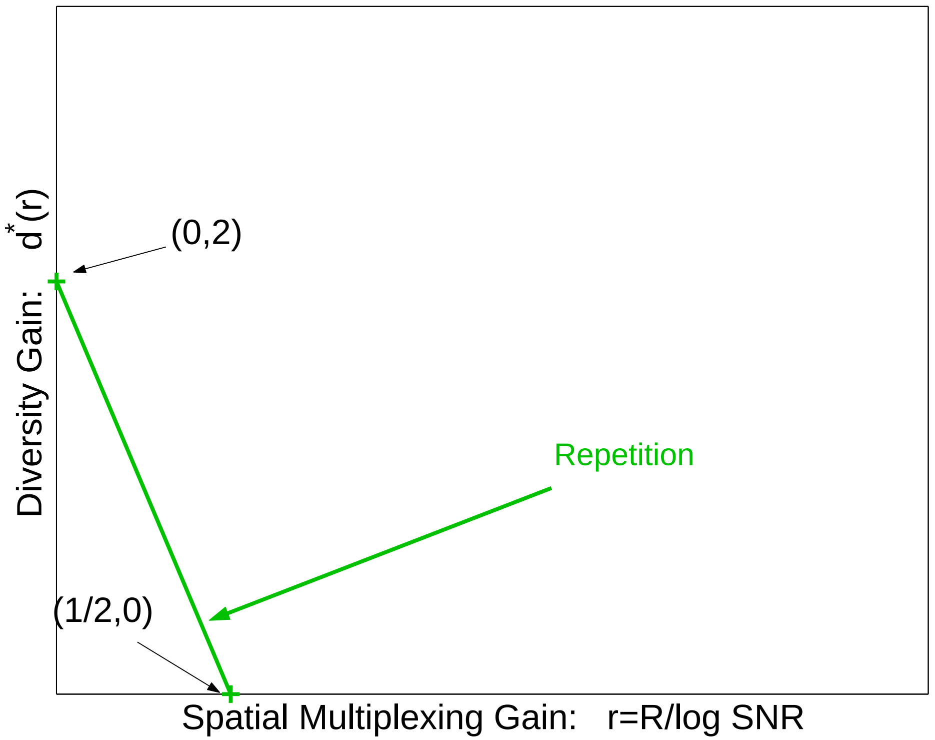

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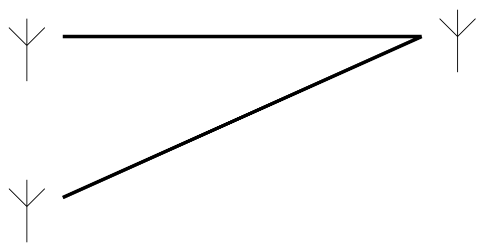

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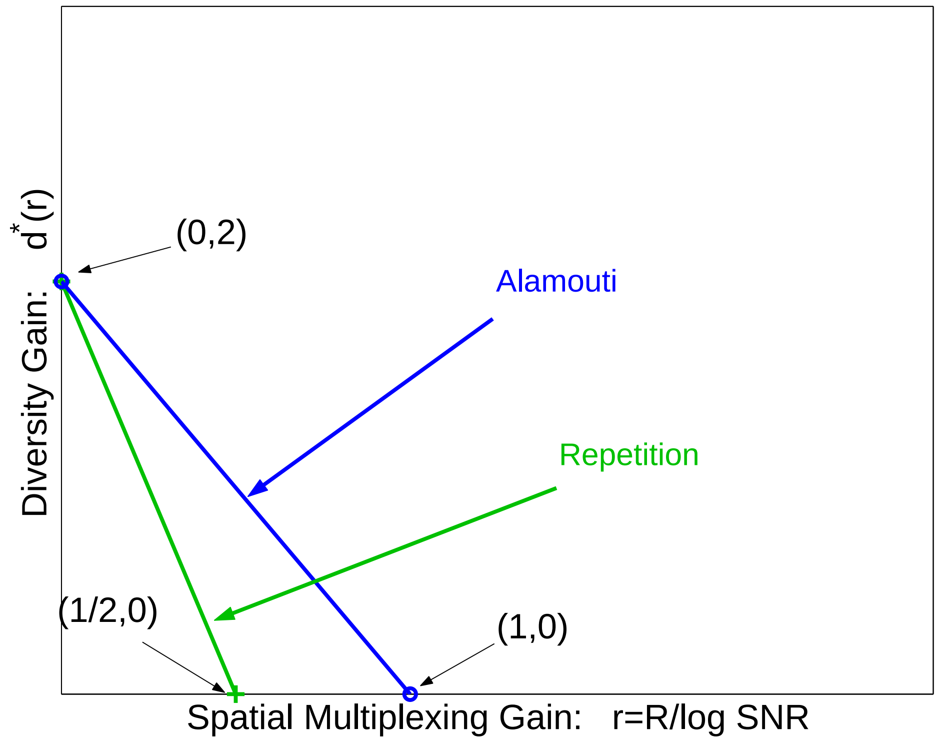

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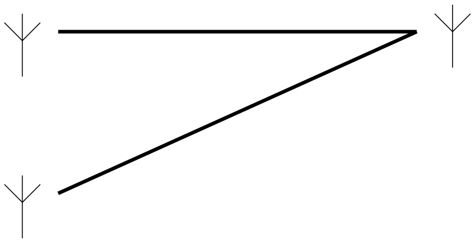
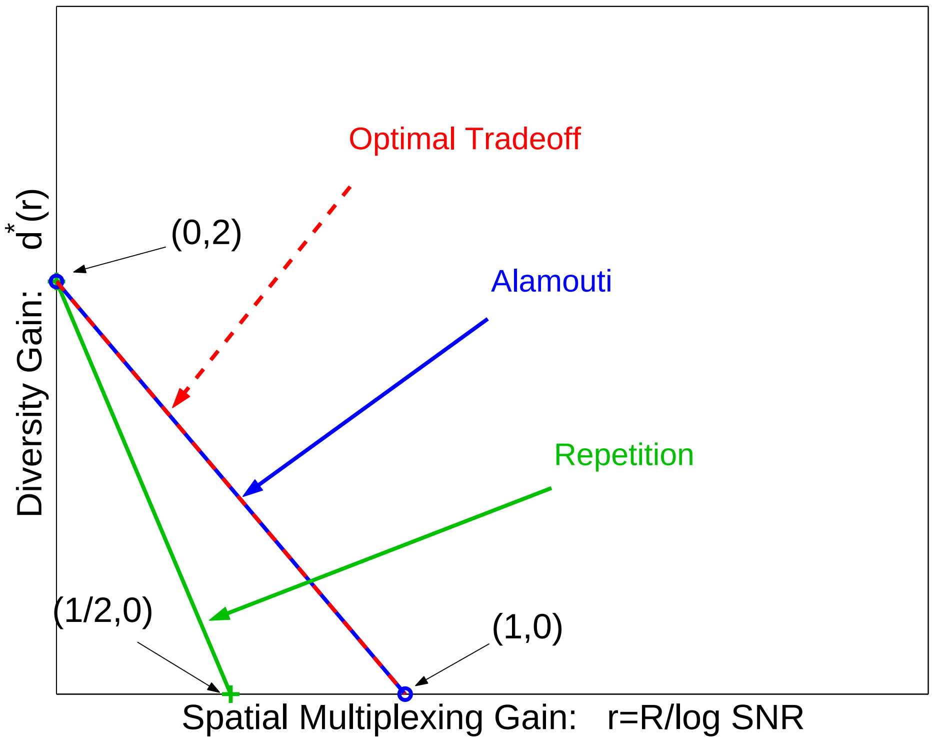
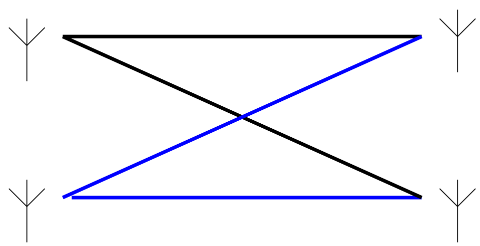


Repetition: $\mathbf{y}_1 = \|\mathbf{H}\|\mathbf{x}_1 + \mathbf{w}$

Alamouti: $[\mathbf{y}_1 \mathbf{y}_2] = \|\mathbf{H}\|[\mathbf{x}_1 \mathbf{x}_2] + [\mathbf{w}_1 \mathbf{w}_2]$



Comparison: 2×2 System



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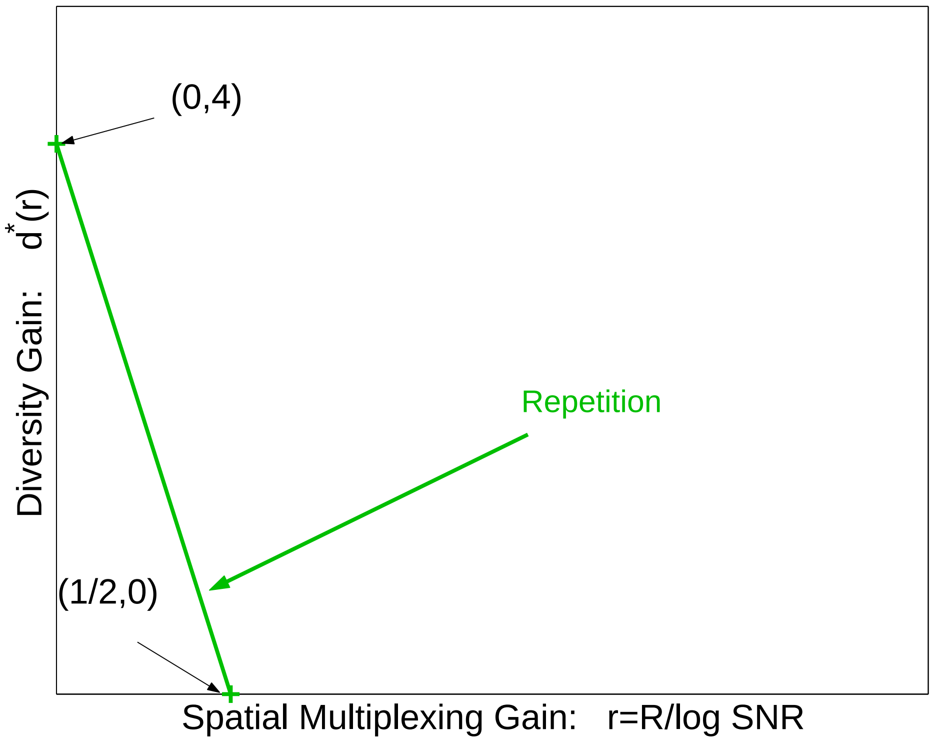

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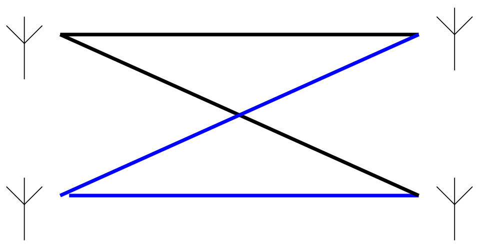

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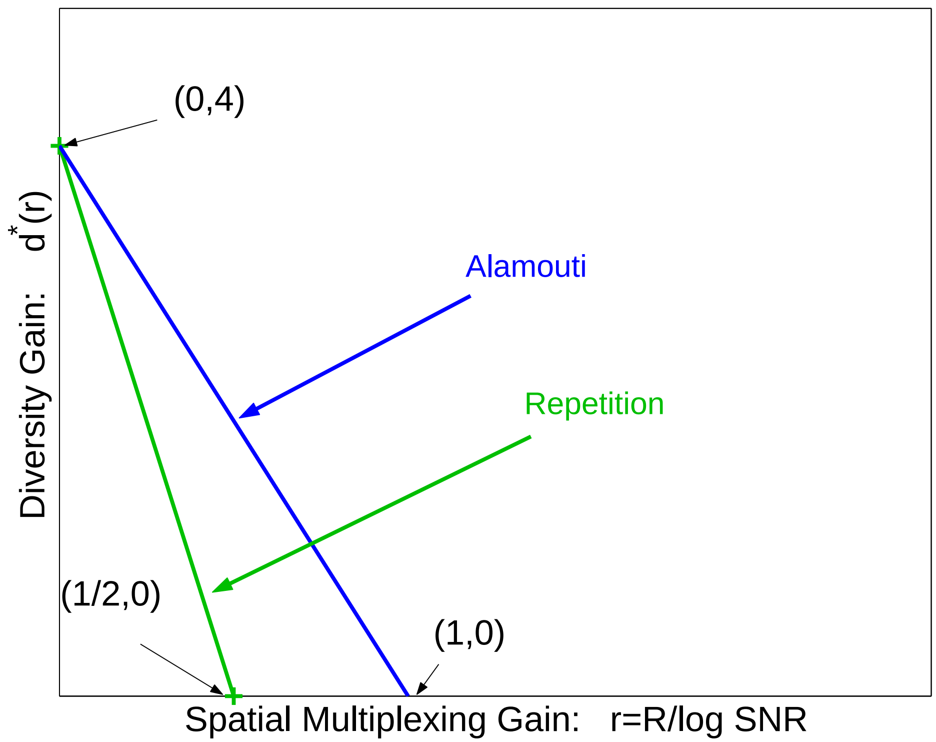

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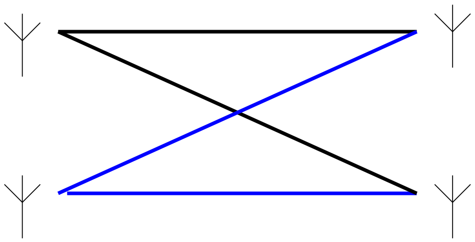

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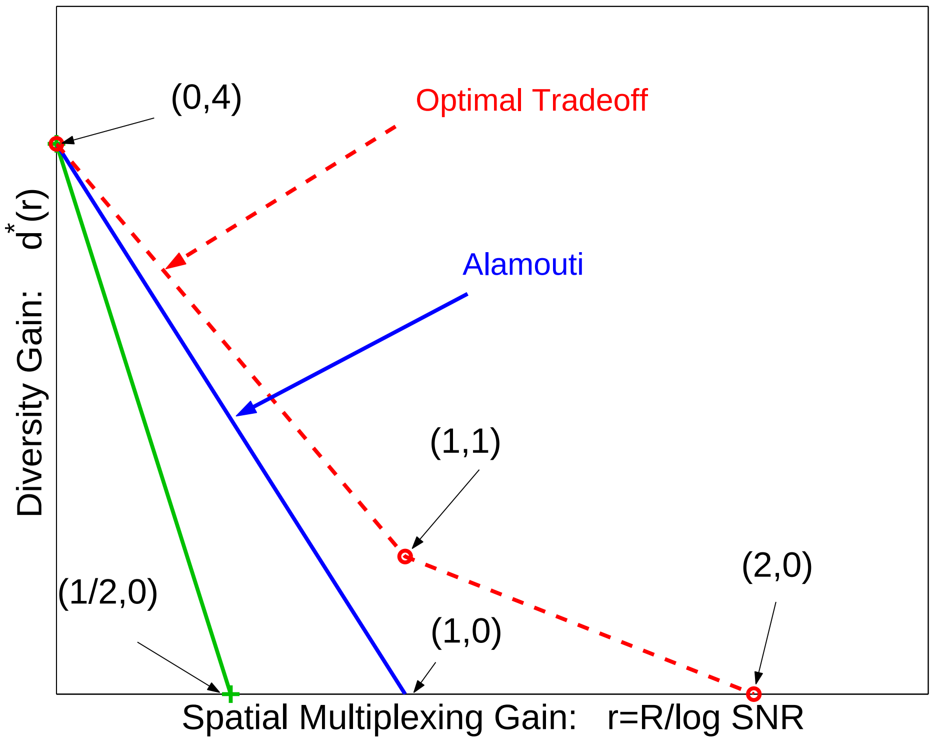
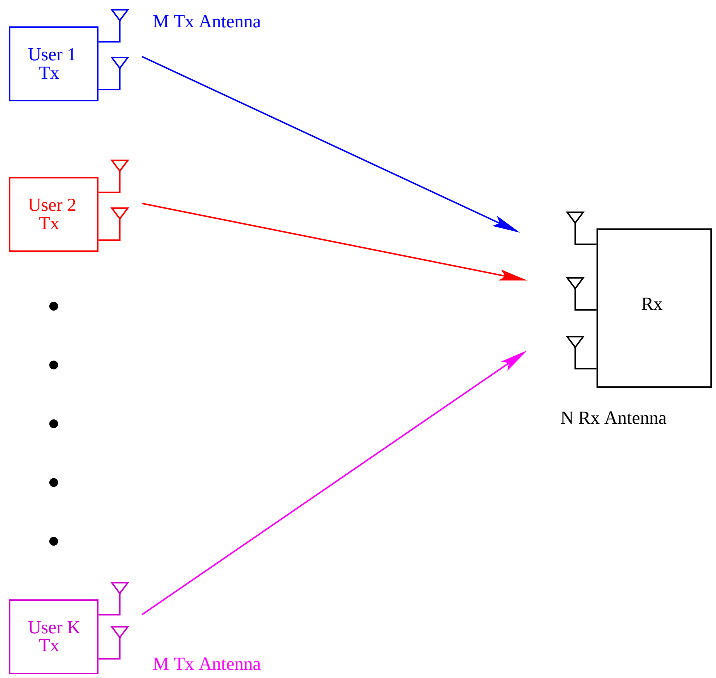


Talk Outline

- point-to-point MIMO channels
- **multiple access MIMO channels**
- cooperative relaying systems

Multiple Access



In a point-to-point link, multiple antennas provide diversity and multiplexing gain.

In a system with K users, multiple antennas can be used to discriminate signals from different users too.

Continue assuming i.i.d. Rayleigh fading, n receive antennas, m transmit antennas per user.

Multiuser Diversity-Multiplexing Tradeoff

Suppose we want every user to achieve an error probability:

$$P_e \sim \mathsf{SNR}^{-d}$$

and a data rate

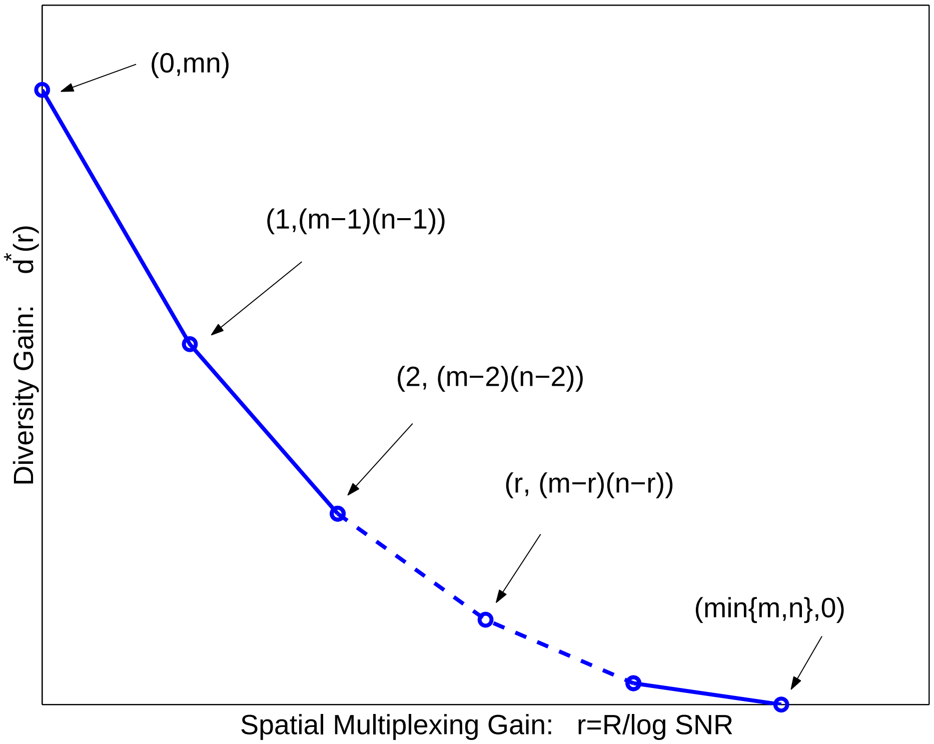
$$R = r \log \mathsf{SNR} \quad \text{bits/s/Hz.}$$

What is the optimal tradeoff between the diversity gain d and the multiplexing gain r?

Assume a coding block length $l \geq Km + n - 1$.

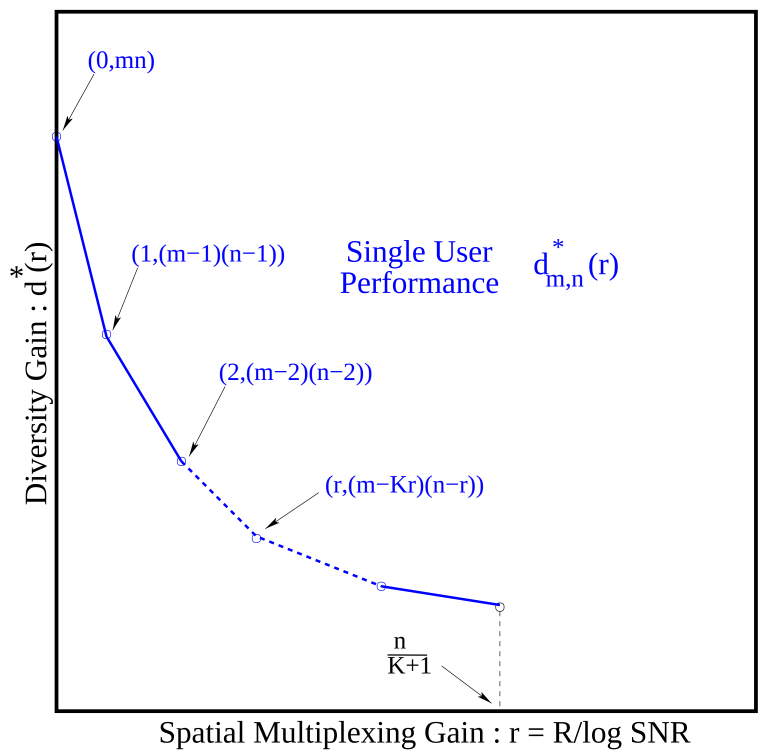
Optimal Multiuser D-M Tradeoff: $m \leq n/(K+1)$

(Tse, Viswanath and Zheng 02)



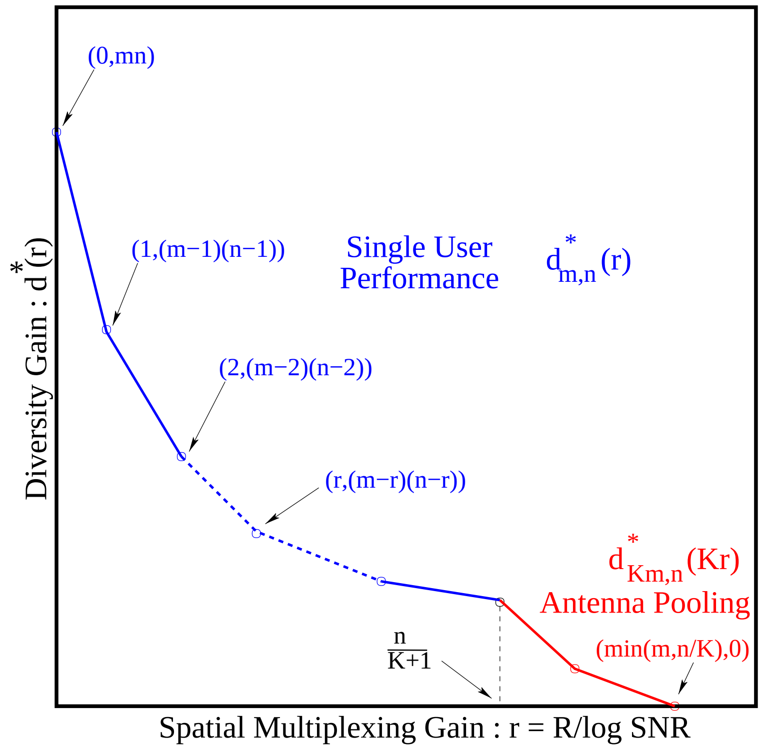
In this regime, diversity-multiplexing tradeoff of each user is as though it is the only user in the system, i.e. $d^*_{m,n}(r)$

Multiuser Tradeoff: $m > n/(K+1)$



Single-user diversity-multiplexing tradeoff up to $r^* = n/(K+1)$.

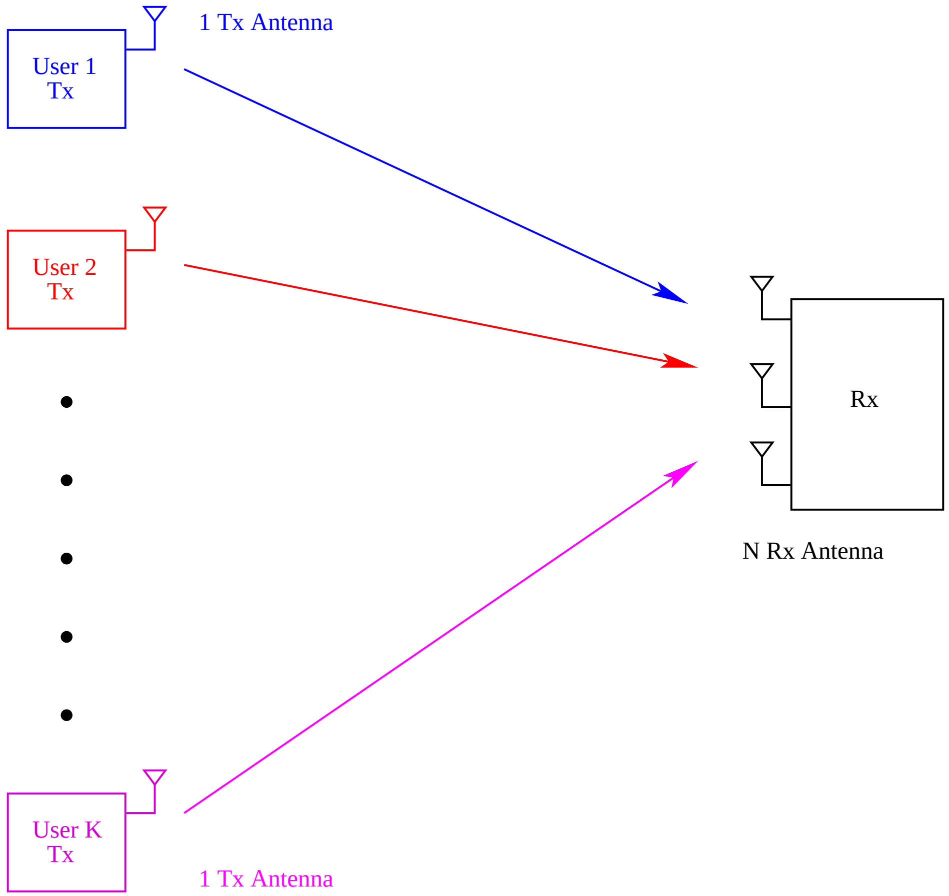
Multiuser Tradeoff: $m > n/(K+1)$



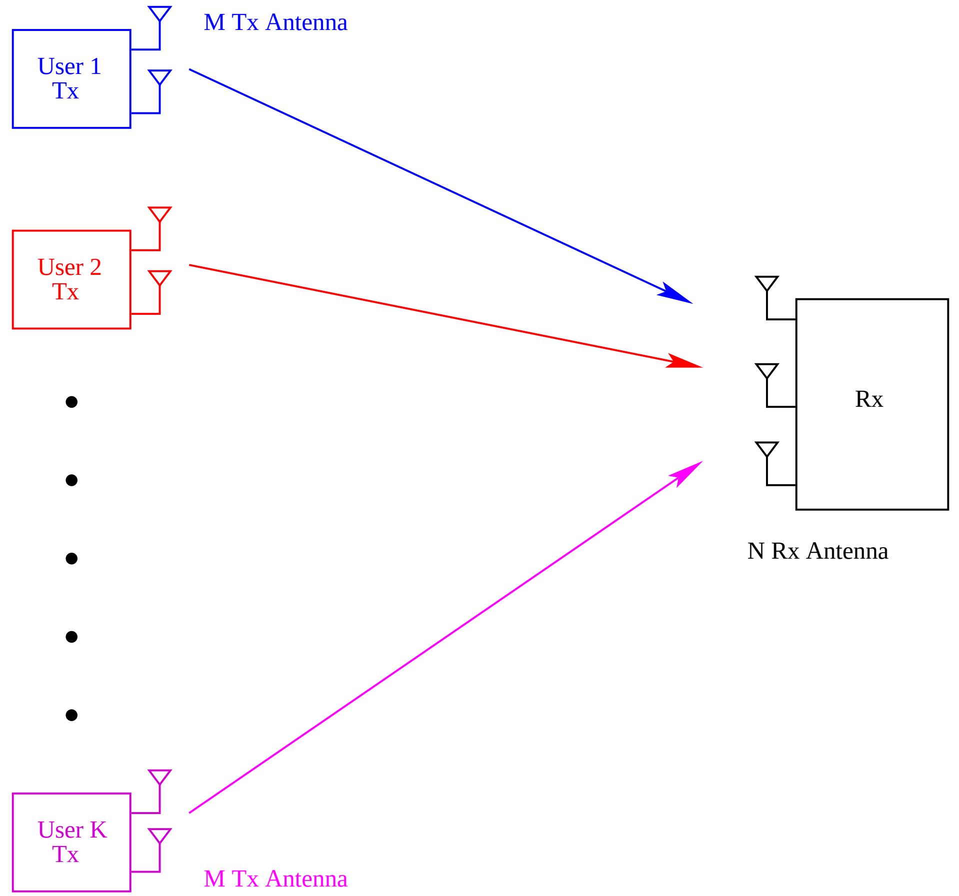
Single-user diversity-multiplexing tradeoff up to $r^* = m/(K+1)$.

For r from $n/(K+1)$ to $\min\{n/K, m\}$, tradeoff is as though the K users are pooled together into a single user with Km antennas and rate Kr, i.e. $d^*_{Km,n}(Kr)$.

Benefit of Dual Transmit Antennas



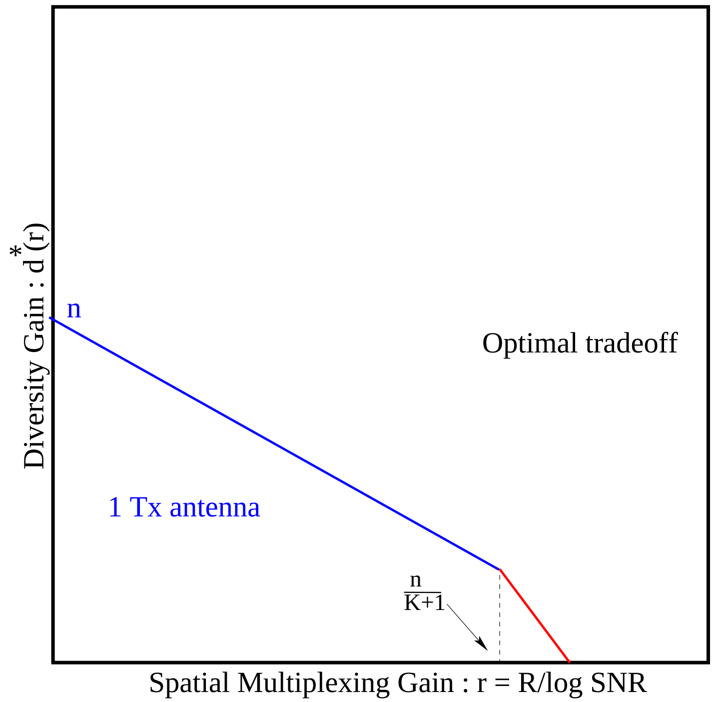
Benefit of Dual Transmit Antennas



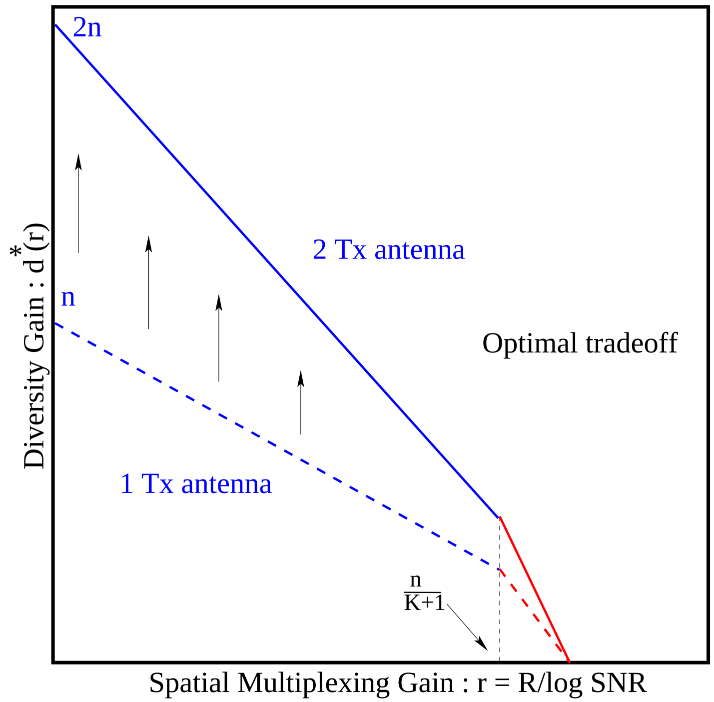
Question: what does adding one more antenna at each mobile buy me?

Assume there are more users than receive antennas.

Answer


Diversity Gain : d*(r)
n
Optimal tradeoff
1 Tx antenna
n/(K+1)
Spatial Multiplexing Gain : r = R/log SNR

Answer

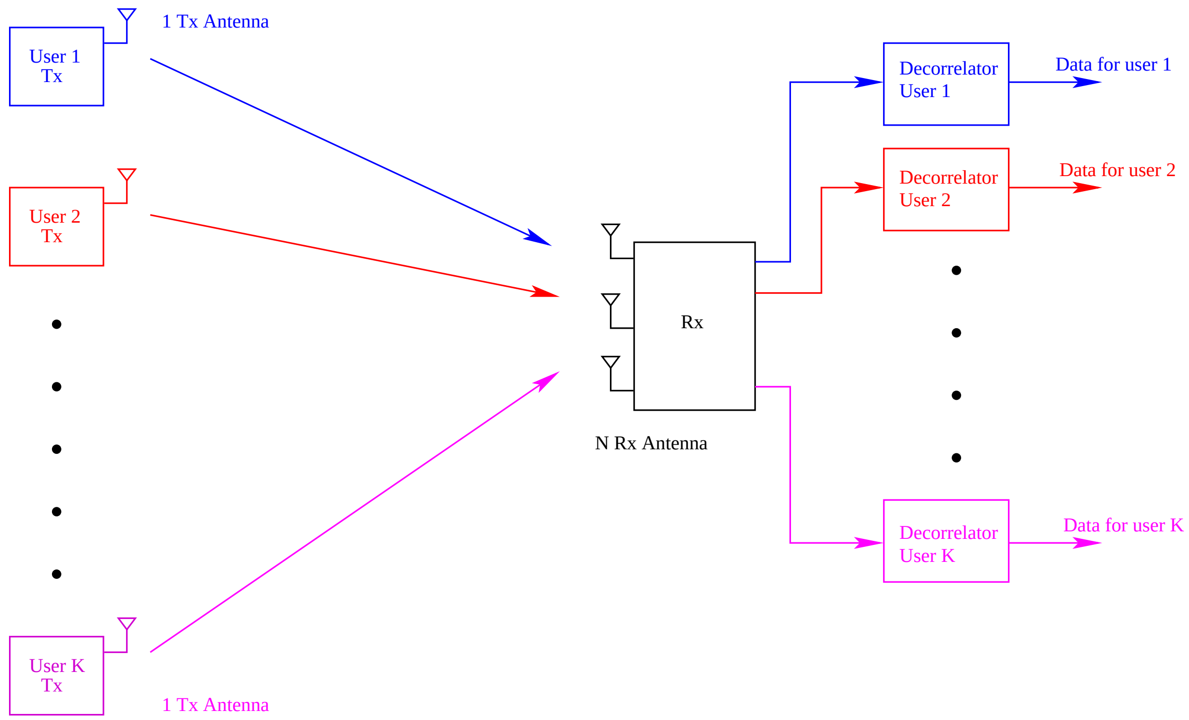


Adding one more transmit antenna does not increase the number of degrees of freedom for each user.

However, it increases the maximum diversity gain from n to $2n$.

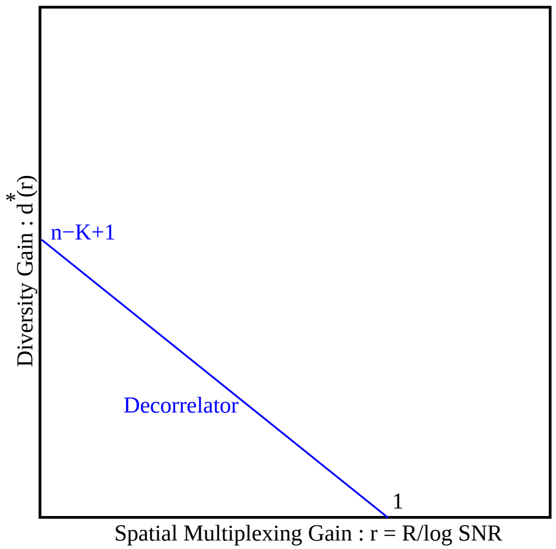
More generally, it improves the diversity gain $d(r)$ for every r.

Suboptimal Receiver: the Decorrelator/Nuller



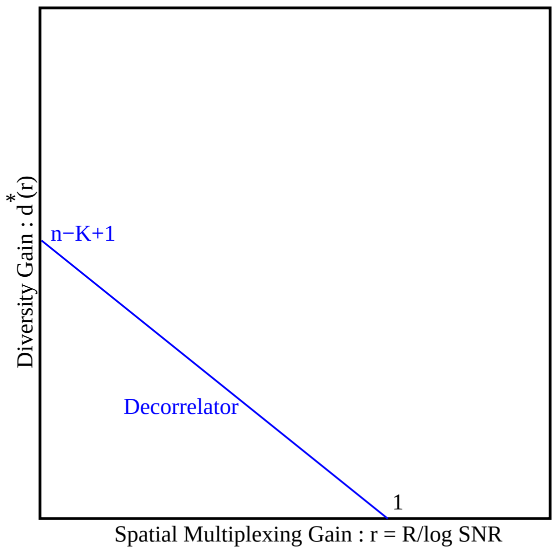
Consider only the case of $m = 1$ transmit antenna for each user and number of users $K < n$.

Tradeoff for the Decorrelator



Maximum diversity gain is $n - K + 1$: "costs $K - 1$ diversity gain to null out $K - 1$ interferers." (Winters, Salz and Gitlin 93)

Tradeoff for the Decorrelator



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Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.

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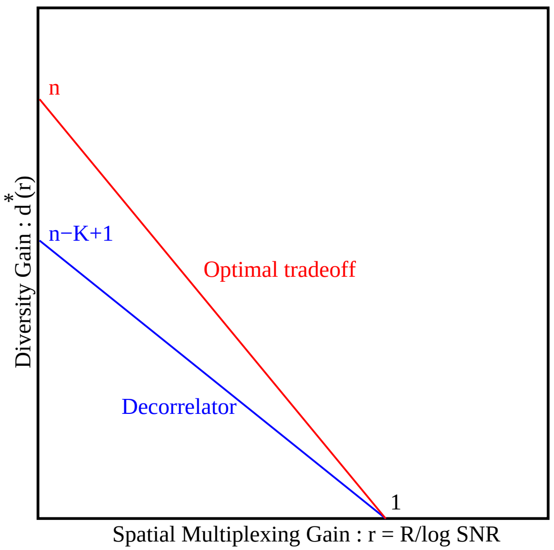

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Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.

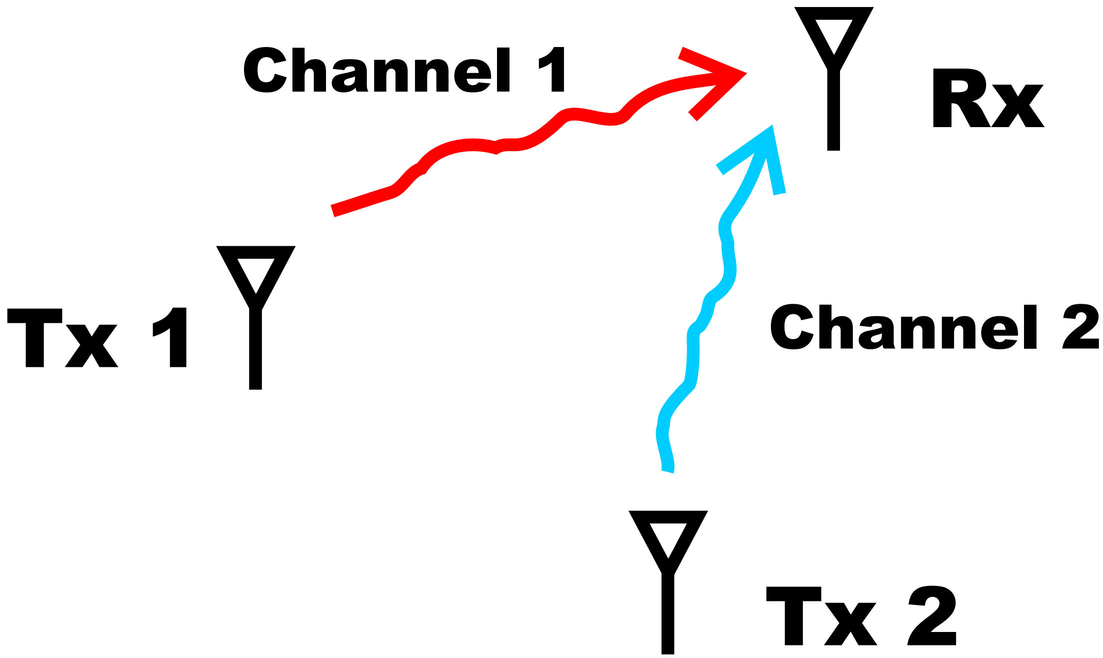
Optimal tradeoff curve is also a straight line but with a maximum diversity gain of n.

Adding one receive antenna provides more reliability per user and accommodate 1 more user.

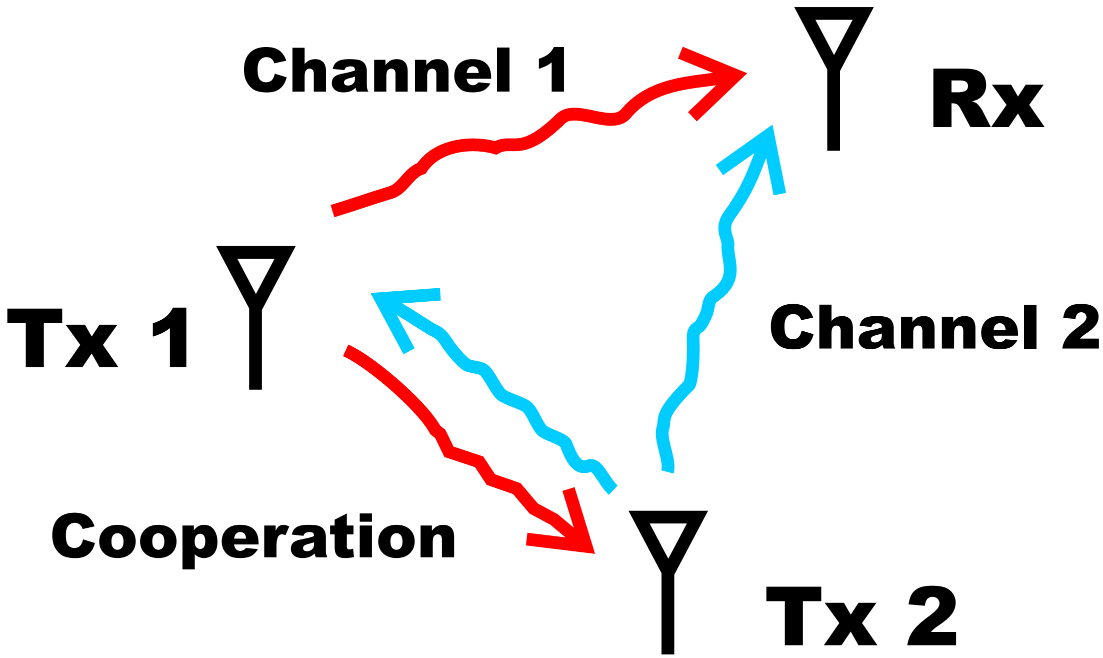
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- **cooperative relaying systems**

Cooperative Relaying



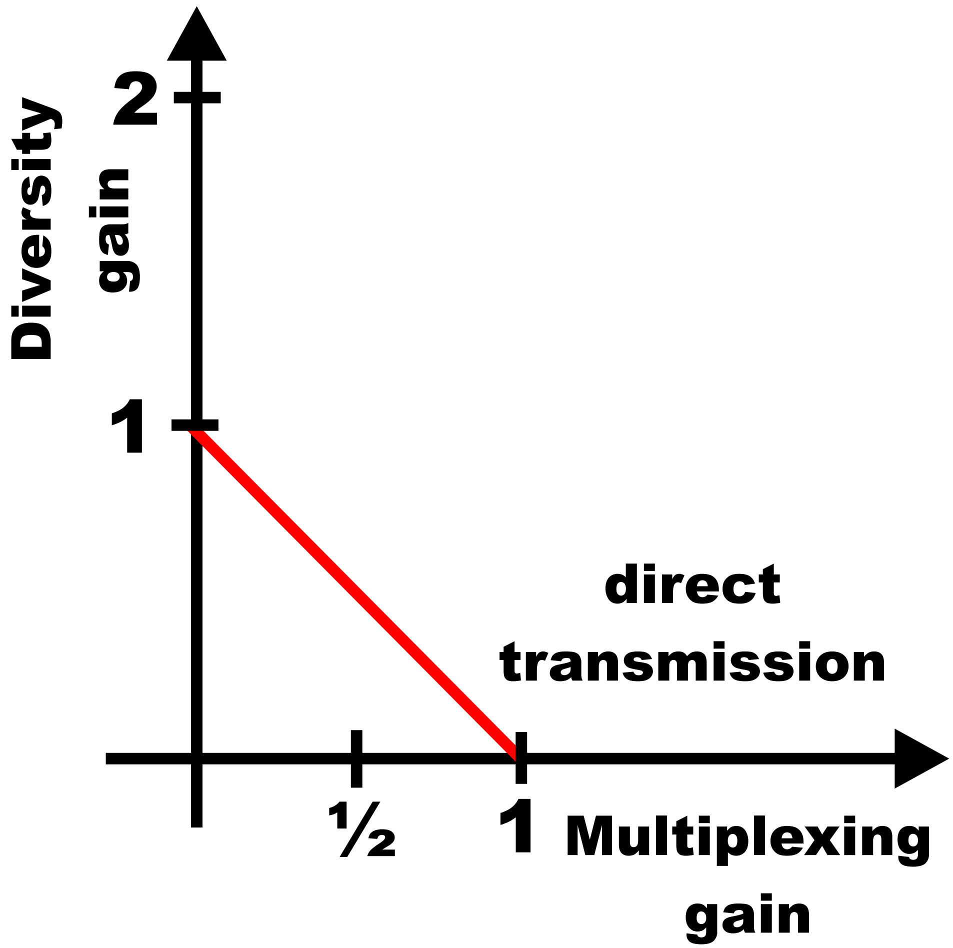
Cooperative Relaying



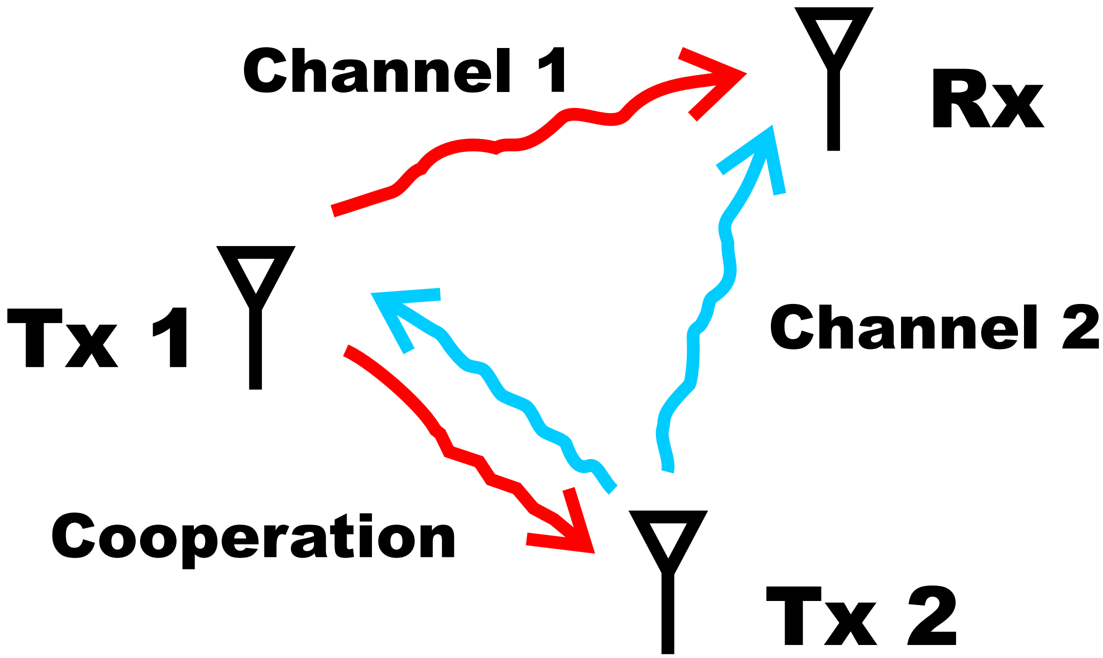
Cooperative relaying protocols can be designed via a diversity-multiplexing tradeoff analysis.

(Laneman, Tse and Wornell 01)

Tradeoff Curves of Relaying Strategies



Cooperative Relaying



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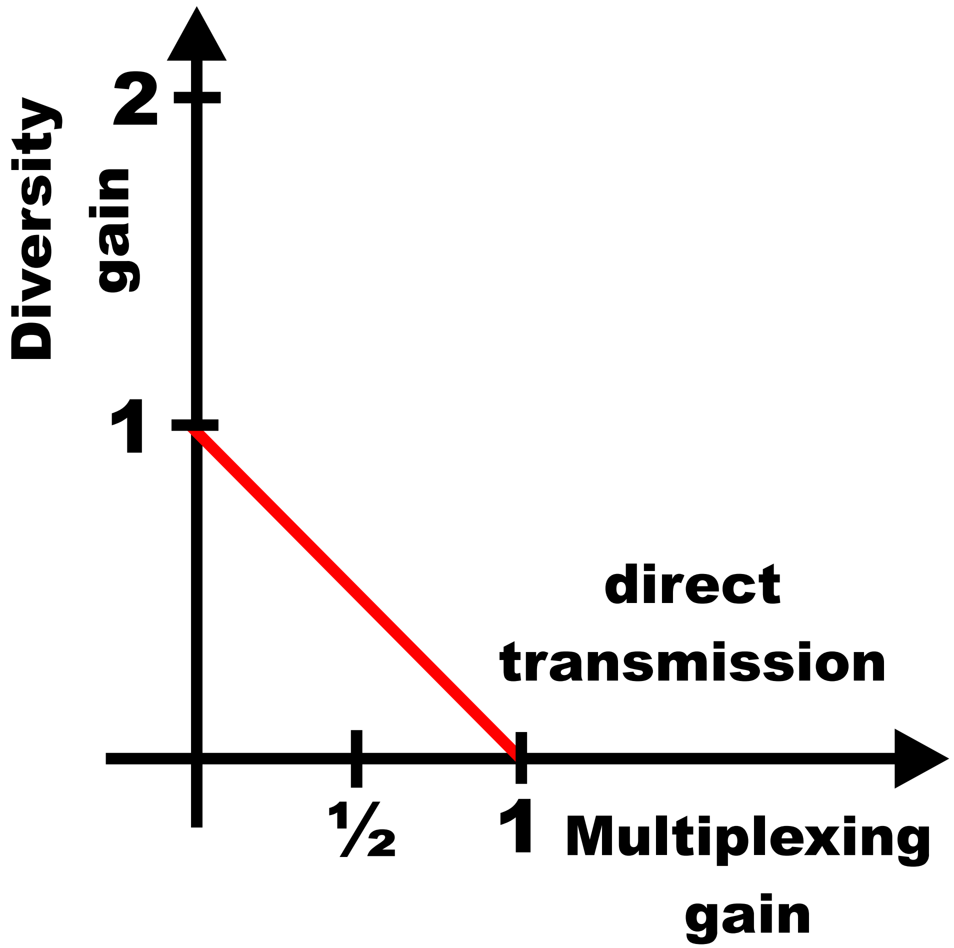
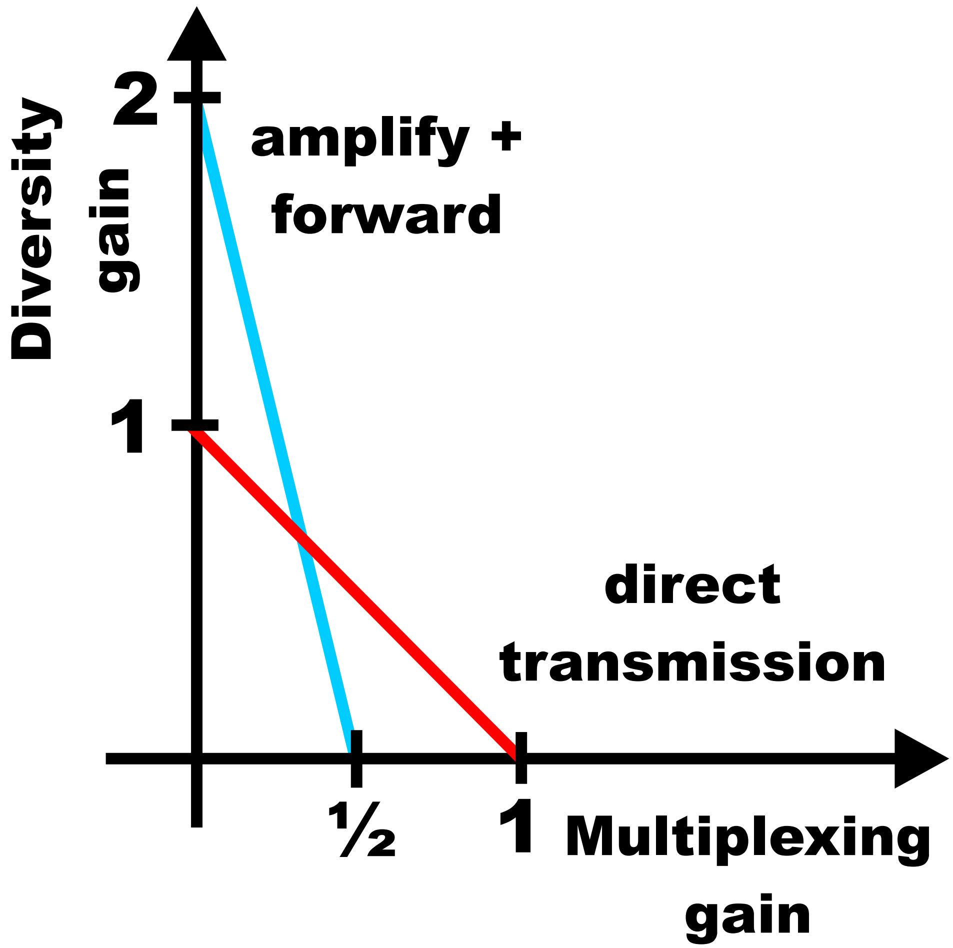


Tradeoff Curves of Relaying Strategies



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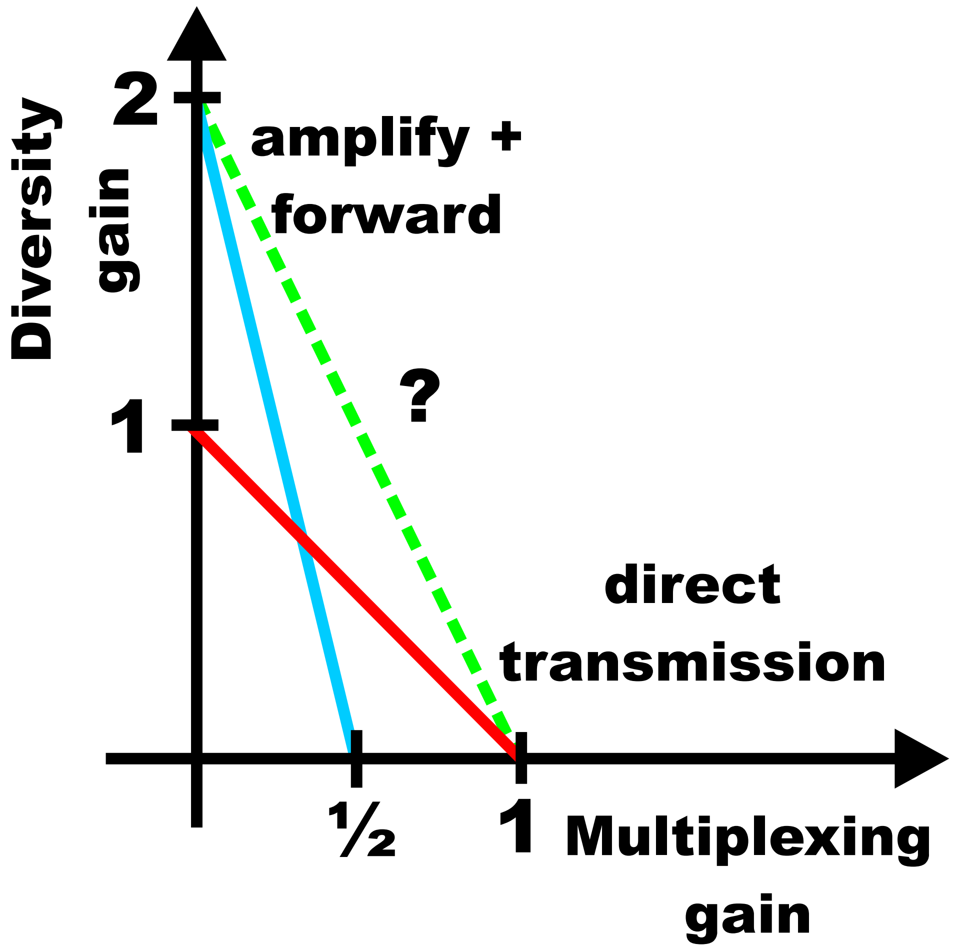

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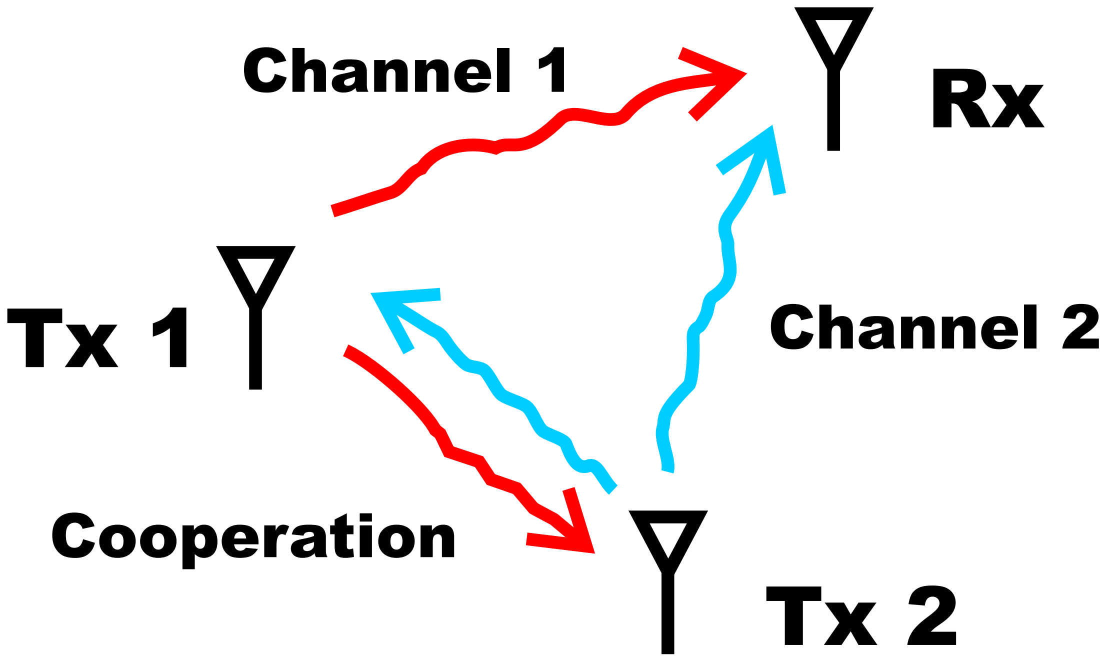

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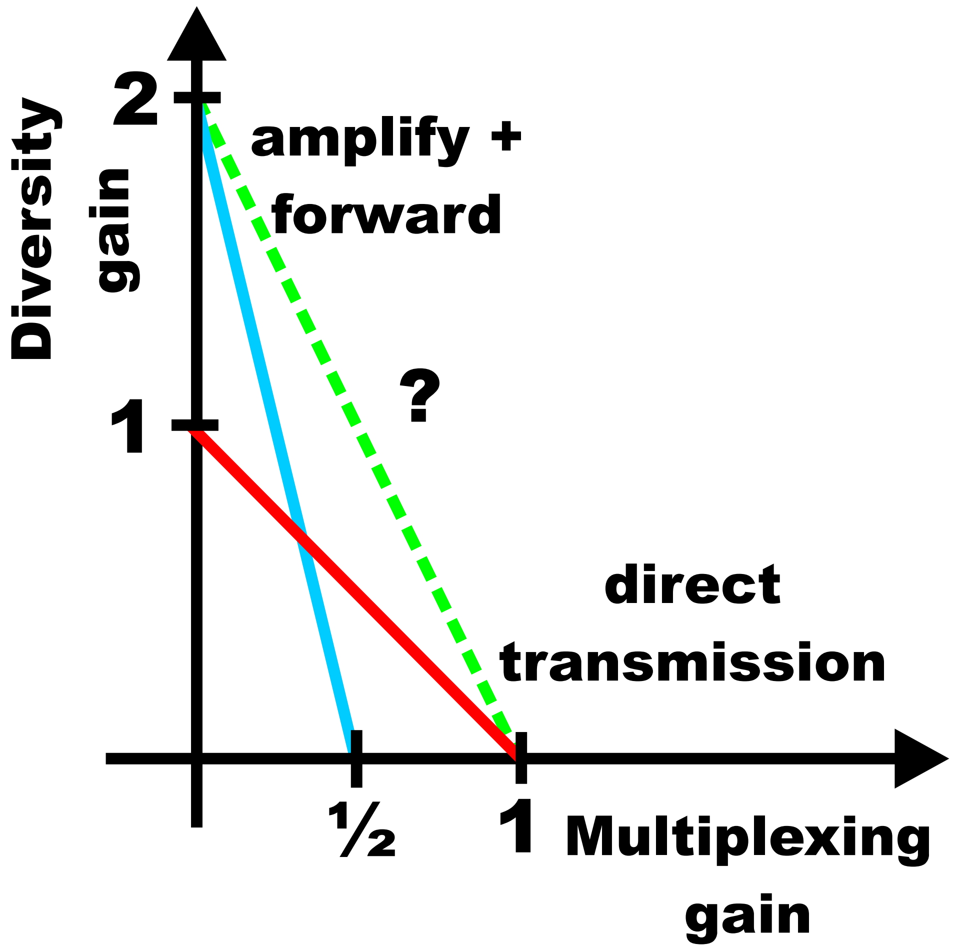

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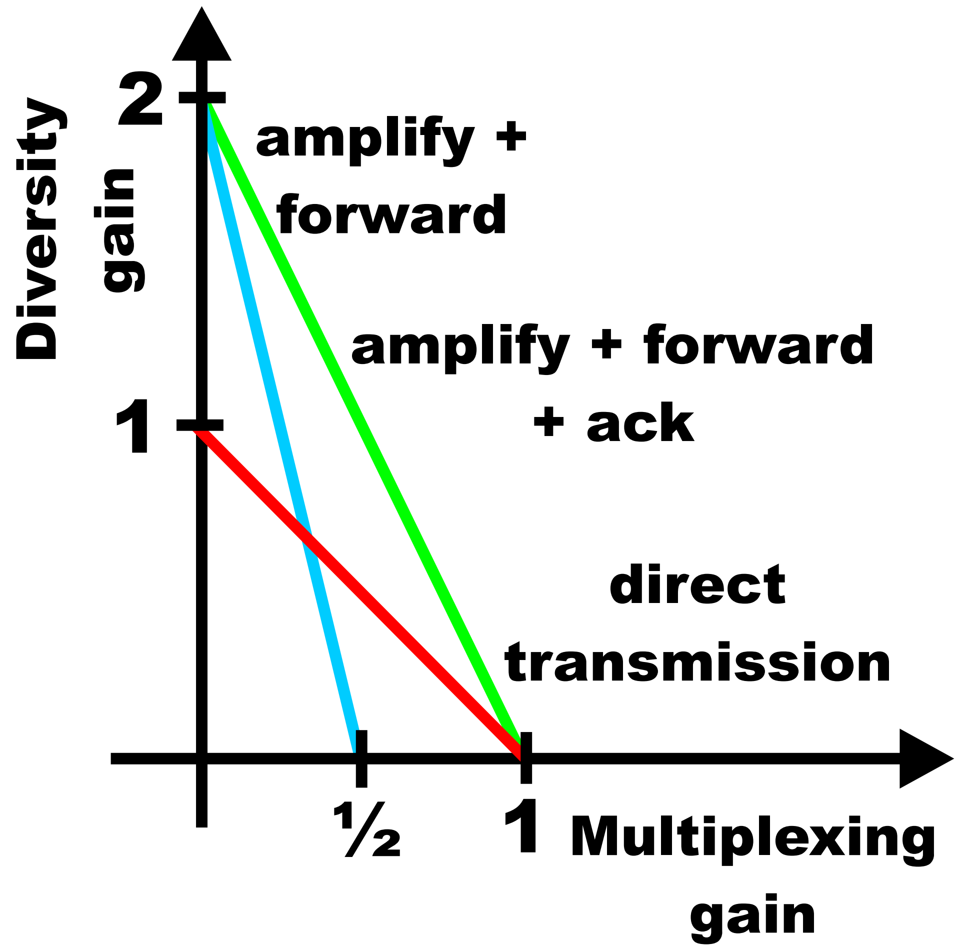

Conclusion

Diversity-multiplexing tradeoff is a unified way to look at performance over wireless channels.

Future work:

- Code design.
- Application to other wireless scenarios.
- Extension to channel-uncertainty-limited rather than noise-limited regime.