THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:559 **Spectrum Management and Wireless Systems** **Fall 2003**

Problem Set 1

1. **Basic Grazing:** Let the amount of grazing done by herd i be g_i and the benefit reaped by shepherd i be $B_i = B(Q(\sum_i g_i), g_i)$ where $Q()$ is a pasture quality function. As always we assume that $Q()$ is strictly monotone decreasing in its (non-negative) argument and that

$$\frac{\partial B}{\partial g} = B_g > 0$$

and

$$\frac{\partial B}{\partial Q} = B_Q > 0$$

Then define the aggregate (social) benefit

$$\bar{B} = \sum_j B_j = \sum_j B(Q(\sum_i g_i), g_j)$$

In what follows, assume two herds but generalize to multiple herds if you'd like.

   (a) Please show that greedy optimization of grazing levels g_j by each user always results in suboptimal grazing from the perspective of an optimum sharing solution where $\bar{B}$ is maximized by considering the $\{g_j\}$ jointly.

   (b) This result is called the "tragedy of the commons." Solve for both the greedy optima and the aggregate (social) otima when we have $Q(g) = e^{-g}$ and $B(Q,g) = Qg$. Exactly how tragic is this tragedy (how much more poorly does greedy optimization perform than joint optimization)?

   (c) In general, do we still have a tragedy if the performance measures from the perspective of each user are different: i.e., different $Q^{(j)}$ and $B^{(j)}$ for each user j?

   (d) The tragedy is often billed as a collapse of the grazing infrastructure. See if you can find general properties of $Q()$ and $B()$ which lead to collapse, within the constraints provided in the problem description.

2. **Dangerous Grazing Games:** Now suppose that each shepherd can adjust the size of the other herd through some means (high power rifle, cow/sheep/goatnapping). Model this adjustment as $\tilde{g}_2 = \alpha_{12} g_2$ and $\tilde{g}_1 = \alpha_{21} g_1$ where α_{ij} is the scaling factor imposed on herd j by shepherd i, $i \neq j$, $\alpha_{ij} \in [0, \infty)$. You may again assume only two herds, but generalize to multiple herds as you see fit.

   (a) Show that if each shepherd i has control over both α_{ij} and g_i that greedy optimization without regard for the actions of others must lead to total collapse.

(b) Now suppose symmetric action so that each shepherd knows that whatever $\alpha_{ij} = \alpha$ they use, the same α will we used by other shepherds. Assume identical quality and benefit functions for each herd. Assume each grazing constant g_i fixed and show that greedy and fair optimization leads to a social optimum. What about greedy but unfair optimization?

(c) Now suppose symmetric action but asymmetric control over grazing g_i. Compare greedy and social optima.

(d) Now think about dynamic control of the individual α_{ij}. Suppose there are limits to how rapidly the α_{ij} can be changed. Might symmetric action arise in practice or no?

(e) Suppose guns and kidnapping are outlawed. Might symmetric action arise naturally to control grazing. Assume identical cost functions for each herd.

3. **Stupid Math Tricks:**

A function $f()$ is convex if $\forall x_0, x_1 \in \mathcal{D}_f$ and $\lambda \in [0,1]$ we have

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda) f(x_1)$$

with equality iff $\lambda = 0,1$ or $x_0 = x_1$. Prove that this definition is equivalent to the usual $\frac{d^2 f(x)}{dx^2} \geq 0$ for a single variable function.

HINT: For any twice differentiable function $f(x)$ in an interval (x_0, x_1) we have

$$f(x) = f(x_0) + \left(\frac{df}{dx}|_{x_0}\right)(x - x_0) + \frac{1}{2}\left(\frac{d^2 f}{dx^2}|_{x_0}\right)(\xi - x_0)^2$$