Hi Folks,
Manasi asked me a question and though I made noises in the right direction, I pretty much screwed up in what I wrote down in an email. This will serve to fix my errors and be an indirect apology to Manasi.

Suppose you have a set of identically distributed random variables \( K_i \) where each is geometric with parameter \( p \). That is

\[
p_{K_i}(k) = p(1-p)^{k-1}
\]

for \( k = 1, 2, \ldots \). We can think of \( K_i \) as the number of trials up to and including a first “success.”

Now, suppose we run the same “success” experiment independently \( r \) times. That is, we’re looking for the number of trials up to and including the \( r \)th success. Then we have a new random variable \( K = K_1 + K_2 + \cdots + K_r \). Since the \( K_i \) are independent, the distribution of the sum is the convolution of the distributions (A VERY IMPORTANT FACT! HINT HINT!!!).

However, unless you’re a masochist and like doing convolution as opposed to multiplication, the speediest way to figure out the distribution on \( K \) is to go to “frequency domain” and use moment generating functions. So, the moment generating function for \( K_i \) is

\[
\varphi_{K_i}(s) = E[e^{sk}] = \sum_{k=1}^{\infty} p(1-p)^{k-1} e^{sk} = \frac{pe^s}{1 - (1-p)e^s}
\]

which immediately means that

\[
\varphi_K(s) = \left( \frac{pe^s}{1 - (1-p)e^s} \right)^r = p^r e^{rs} \left( \frac{1}{1 - (1-p)e^s} \right)^r
\]
And I dare you to try to take the inverse transform directly to get back to \( p_K(k) \).
(I have a personal dislike of doing contour integrations, so inverse \( Z \) transforms and inverse Laplace transforms often bedevil me. Luckily, I have lots of company in my laziness.)

When I’m confronted with something I don’t know (or don’t want to do using brute force), I nose around for an easy way out. Looking at the fraction raised to a power makes me immediately think of

\[
\frac{d}{d\theta} \frac{1}{1-\theta} = \left( \frac{1}{1-\theta} \right)^2
\]

Therefore

\[
\frac{d^\ell}{d\theta^\ell} \frac{1}{1-\theta} = \ell! \left( \frac{1}{1-\theta} \right)^{\ell+1}
\]

Using this lovely fact we have (setting \( \theta = (1 - p)e^s \))

\[
\phi_K(s) = p^r e^{sr} \frac{1}{(r-1)!} \frac{d^{r-1}}{d\theta^{r-1}} \frac{1}{1-\theta}
\]

Suddenly, I’m much happier because I immediately remember that

\[
\sum_{m=0}^{\infty} q^m = \frac{1}{1-q}
\]

which leads to

\[
\phi_K(s) = p^r e^{sr} \frac{1}{(r-1)!} \frac{d^{r-1}}{d\theta^{r-1}} \sum_{k=1}^{\infty} \theta^{k-1} = p^r e^{sr} \frac{1}{(r-1)!} \sum_{k=r}^{\infty} \frac{(k-1)!}{(k-1-(r-1))!} \theta^{k-1-(r-1)}
\]

which we simplify as

\[
\phi_K(s) = p^r e^{sr} \frac{1}{(r-1)!} \sum_{k=r}^{\infty} \frac{(k-1)!}{(k-r)!} \theta^{k-r}
\]

We then substitute for \( \theta \) to obtain

\[
\phi_K(s) = \sum_{k=r}^{\infty} p^r e^{sr} \frac{1}{(r-1)! (k-r)!} (1-p)^{k-r} e^{s(k-r)}
\]
and rearrange a little to get

\[ \phi_K(s) = \sum_{k=r}^{\infty} e^{sk} \binom{k - 1}{r - 1} p^r (1 - p)^{k-r} \]

Hmmmmm! This looks just like \( E[e^{sk}] \) if

\[ p_K(k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r} \]

\( k = r, r + 1, \ldots \). AND WE’RE DONE since that’s the Pascal distribution!

Laziness has its rewards!