332:541   Stochastic Signals and Systems   Fall 2001
MIDTERM EXAMINATION

You have 1.5 hours to answer the following three questions. You are allowed both sides of an 11x8.5 inch piece of paper for notes. GOOD LUCK!

1. *(60 points)* **Gaussian Properties:**

   X and Y are zero mean jointly Gaussian random variables with correlation coefficient ρ and unit variances. The joint PDF is

   $$f_{XY}(x,y) = \frac{1}{2\pi}\sqrt{\frac{1}{1-\rho^2}}\, e^{\frac{-x^2-y^2+2xy\rho}{2(1-\rho^2)}}$$

   (a) *(10 points)* Suppose $\rho = 0$ and $W = aX$ where $a \in \Re$. What is the PDF of W?

   (b) *(10 points)* Suppose $\rho = 0$ and $Z = aX + bY$ where $a, b \in \Re$. PROVE that Z is Gaussian. What is the expected value of Z? What is its variance?

   (c) *(10 points)* For general ρ, PROVE that the PDF for $Z = aX + bY$ is Gaussian where $a, b \in \Re$.

   (d) *(10 points)* For general ρ, suppose $W = cX + dY$ and $Z = aX + bY$. PROVE that Z and W are JOINTLY Gaussian for $a, b, c, d \in \Re$.

   (e) *(10 points)* Assume the result of the previous part. Can you use this result to show that

   $$Z_j = \sum_{i=1}^{N} a_{ij} X_i$$

   are jointly Gaussian random variables if the X_i are jointly Gaussian and the $a_i \in \Re$?

   (f) *(10 points)* For $W = cX + dY$ and $Z = aX + bY$, assume that W and Z are jointly Gaussian and find a relationship between a, b, d and d which makes W and Z INDEPENDENT.

2. *(50 points)* **Theoretical Stuff:**

   (a) *(10 points)* Give an example of a PDF for a non-negative continuous random variable X such that $E[X]$ does not exist.

   (b) *(10 points)* PROVE/DISPROVE: for a non-negative continuous random variable X,

   $$E[X] = \int_0^\infty (1 - F_X(x))dx$$

   where $F_X()$ is the CDF of X. Then find a looser set of restrictions on X if possible.

   HINT: You may assume that the integral above exists – which says something about how fast $1 - F(x)$ goes to zero in x.

(c) *(10 points)* PROVE/DISPROVE: for a convex function $g()$

$$E[g(X)] \geq g(E[X])$$

where X is a continuous random variable for which $E[X]$ exists.

HINT: See if you can FORMALLY use the discrete Jensen's inequality in a limiting way. Be careful or I'll be vicious in grading.

(d) *(10 points)* We define random variables $X = \cos\Theta$ and $Y = \sin\Theta$ where Θ is a uniform random variable on $[-\pi, \pi]$. What are $E[X]$ and $E[Y]$? What is $E[XY]$? Are X and Y independent (why/why not)?

(e) *(10 points)* Suppose X and Y are arbitrary zero mean random variables for which $E[XY] = 1$. What can we say (if anything) about the relationship between X and Y?

3. *(50 points)* **Rutgera and the Country Road:** Rutgera Univera, the world famous Rutgers ECE graduate student has bought a house along a single lane country road. Her feet are still sore from her days on a South Pacific island where she was stranded and had to run after small animals for food. So Rutgera walks very slowly and is not very agile.

Rutgera needs to use the road, a very strange road in that it bursts into flames randomly. Rutgera uses her old training in probability and notices that the time τ_i in seconds between successive road flamings i and $i+1$ is an exponential random variable with parameter λ (units 1/seconds) and that these times τ_i are mutually independent.

(a) *(10 points)* Suppose that a bear is walking down the road at speed V just after the last flaming. Each time the road flames, its speed decreases by a factor of 2. Write down an expression for the distance it travels (it can be burned infinitely many times) and find its expected value and variance.

(b) *(10 points)* Rutgera is TOUGH, and being burned is a minor annoyance – like a mosquito bite – so she does not slow down.

So, suppose it takes Rutgera T seconds to CROSS the road independent of whether she gets burned.

If Rutgera starts right after the last flaming, what is the probability that Rutgera does not get burned? Does your answer change depending on the time Rutgera gets started after a burning? Why/why not?

(c) *(10 points)* What is the probability that Rutgera is burned exactly once?

(d) *(10 points)* What is the probability that Rutgera is burned exactly twice?

(e) *(10 points)* What is the probability that Rutgera is burned exactly K times?