# Probability and Stochastic Processes:
# A Friendly Introduction for Electrical and Computer Engineers

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**Problem Solutions** : Yates and Goodman,8.1.3 8.2.4 8.2.5 8.2.7 8.2.8 8.3.3 8.4.2 8.4.3 and 8.4.4

**Problem 8.1.3**

$X_1, X_2 \ldots X_n$ are independent uniform random variables with mean value $\mu_X = 7$ and $\sigma_X^2 = 3$

(a) Since $X_1$ is a uniform random variable, it must have a uniform PDF over an interval $[a, b]$. From Appendix A, we can look up that $\mu_X = (a+b)/2$ and that $\mathrm{Var}[X] = (b-a)^2/12$. Hence, given the mean and variance, we obtain the following equations for $a$ and $b$.

$$(b-a)^2/12 = 3 \qquad (a+b)/2 = 7$$

Solving these equations yields $a = 4$ and $b = 10$ from which we can state the distribution of $X$.

$$f_X(x) = \begin{cases} 1/6 & 4 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

(b) From Theorem 8.1, we know that

$$\mathrm{Var}[M_{16}(X)] = \frac{\mathrm{Var}[X]}{16} = \frac{3}{16}$$

(c)

$$P[X_1 \ge 9] = \int_9^\infty f_{X_1}(x)\, dx = \int_9^{10} (1/6)\, dx = 1/6$$

(d) The variance of $M_{16}(X)$ is much less than $\mathrm{Var}[X_1]$. Hence, the PDF of $M_{16}(X)$ should be much more concentrated about $E[X]$ than the PDF of $X_1$. Thus we should expect $P[M_{16}(X) > 9]$ to be much less than $P[X_1 > 9]$.

$$P[M_{16}(X) > 9] = 1 - P[M_{16}(X) \le 9] = 1 - P[(X_1 + \cdots + X_{16}) \le 144]$$

By a Central Limit Theorem approximation,

$$P[M_{16}(X) > 9] \approx 1 - \Phi\left(\frac{144 - 16\mu_X}{\sqrt{16}\sigma_X}\right) = 1 - \Phi(2.66) = 0.0039$$

As we predicted, $P[M_{16}(X) > 9] \ll P[X_1 > 9]$.

**Problem 8.2.4**
The $N[0,1]$ random variable $Z$ has MGF $\phi_Z(s) = e^{s^2/2}$. Hence the Chernoff bound for $Z$ is

$$P[Z \geq c] \leq \min_{s\geq 0} e^{-sc} e^{s^2/2} = \min_{s\geq 0} e^{s^2/2-sc}$$

We can minimize $e^{s^2/2-sc}$ by minimizing the exponent $s^2/2 - sc$. By setting

$$\frac{d}{ds}\left(s^2/2 - sc\right) = 2s - c = 0$$

we obtain $s = c$. At $s = c$, the upper bound is $P[Z \geq c] \leq e^{-c^2/2}$. The table below compares this upper bound to the true probability. Note that for $c = 1, 2$ we use Table 4.1 and the fact that $Q(c) = 1 - \Phi(c)$.

| | $c=1$ | $c=2$ | $c=3$ | $c=4$ | $c=5$ |
|---|---|---|---|---|---|
| Chernoff bound | 0.606 | 0.135 | 0.011 | $3.35\times 10^{-4}$ | $3.73\times 10^{-6}$ |
| $Q(c)$ | 0.1587 | 0.0228 | 0.0013 | $3.17\times 10^{-5}$ | $2.87\times 10^{-7}$ |

We see that in this caase, the Chernoff bound typically overestimates the true probability by roughly a factor of 10.

**Problem 8.2.5**
For an $N[\mu,\sigma^2]$ random variable $X$, we can write

$$P[X \geq c] = P[(X-\mu)/\sigma \geq (c-\mu)/\sigma] = P[Z \geq (c-\mu)/\sigma]$$

Since $Z$ is $N[0,1]$, we can apply the result of Problem 8.2.4 with $c$ replaced by $(c-\mu)/\sigma$. This yields

$$P[X \geq c] = P[Z \geq (c-\mu)/\sigma] \leq e^{-(c-\mu)^2/2\sigma^2}$$

**Problem 8.2.7**
process of rate $\lambda = 1/2$ trains/minute, $N$, the number of trains in the first 30 minutes, is a Poisson random variable with parameter $\alpha = 30\lambda = 15$. The PMF of $N$ is

$$P_N(n) = \begin{cases} 15^n e^{-15}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Since $X$ is the arrival time of the third train, $X \geq 30$ if and only if there were $N \leq 2$ trains in the first 30 minutes. That is,

$$\begin{aligned} P[X \geq 30] = P[N \leq 2] &= P_N(0) + P_N(1) + P_N(2) \\ &= (1 + 15 + 15^2/2!)e^{-15} \\ &= 241e^{-15} = 7.37 \times 10^{-5} \end{aligned}$$

In Quiz 8.2, we found the following bounds:

$$\begin{aligned} &P[X \geq 30] \leq \frac{E[X]}{30} = 0.20 \qquad \text{Markov} \\ &P[X \geq 30] \leq 0.021 \qquad \text{Chebyshev} \\ &P[X \geq 30] \leq 7.68 \times 10^{-4} \end{aligned}$$

Comparing the exact probability to the bounds, we see that the Markov inequaland Chebyshev inequalities were extremely weak. On the other hand, the Chernoff bound worked reasonably well.

**Problem 8.2.8**

Let $W_n = X_1 + \cdots + X_n$. Since $M_n(X) = W_n/n$, we can write

$$P[M_n(X) \geq c] = P[W_n \geq nc]$$

Since $\phi_{W_n}(s) = (\phi_X(s))^n$, applying the Chernoff bound to $W_n$ yields

$$P[W_n \geq nc] \leq \min_{s \geq 0} e^{-snc} \phi_{W_n}(s) = \min_{s \geq 0} \left(e^{-sc}\phi_X(s)\right)^n$$

For $y \geq 0$, $y^n$ is a nondecreasing function of $y$. This implies that the value of $s$ that minimizes $e^{-sc}\phi_X(s)$ also minimizes $(e^{-sc}\phi_X(s))^n$. Hence

$$P[M_n(X) \geq c] = P[W_n \geq nc] \leq \left(\min_{s \geq 0} e^{-sc}\phi_X(s)\right)^n$$

**Problem 8.3.3**

Both questions can be answered using the following equation from Example 8.8:

$$P[|R_n - P[A]| \geq c] \leq \frac{P[A](1-P[A])}{nc^2}$$

The unusual part of this problem is that we are given the true value of $P[A]$. Since $P[A] = 0.01$, we can write

$$P[|R_n - P[A]| \geq c] \leq \frac{0.0099}{nc^2}$$

(a) In this part, we meet the requirement by choosing $c = 0.001$ yielding

$$P[|R_n - P[A]| \geq 0.001] \leq \frac{9900}{n}$$

Thus to have confidence level 0.01, we require that $9900/n \leq 0.01$. This requires $n \geq 990,000$.

(b) In this case, we meet the requirement by choosing $c = 10^{-3}P[A] = 10^{-5}$. This implies

$$P[|R_n - P[A]| \geq c] \leq \frac{P[A](1-P[A])}{nc^2} = \frac{0.0099}{n10^{-10}} = \frac{9.9 \times 10^7}{n}$$

The confidence level 0.01 is met if $9.9 \times 10^7/n = 0.01$ or $n = 9.9 \times 10^9$.

**Problem 8.4.2**

(a) From Theorem 7.2, we have

$$\mathrm{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathrm{Cov}\left[X_i, X_j\right]$$

Note that $\text{Var}[X_i] = \sigma^2$ and for $j > i$, $\sigma_{X_i,X_j} = \sigma^2 a^{j-i}$. This implies

$$\begin{aligned}
\text{Var}[X_1 + \cdots + X_n] &= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a^{j-i} \\
&= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} \left(a + a^2 + \cdots + a^{n-i}\right) \\
&= n\sigma^2 + \frac{2a\sigma^2}{1-a} \sum_{i=1}^{n-1} (1 - a^{n-i}) \\
&= n\sigma^2 + \frac{2a\sigma^2}{1-a}(n-1) - \frac{2a}{1-a}\left(a + a^2 + \cdots + a^{n-1}\right) \\
&= \left(\frac{n(1+a)\sigma^2}{1-a}\right) - \frac{2a\sigma^2}{1-a} - 2\sigma^2\left(\frac{a}{1-a}\right)^2 (1 - a^{n-1})
\end{aligned}$$

Since $a/(1-a)$ and $1 - a^{n-1}$ are both nonnegative,

$$\text{Var}[X_1 + \cdots + X_n] \leq n\sigma^2 \left(\frac{1+a}{1-a}\right)$$

(b) Since the expected value of a sum equals the sum of the expected values,

$$E[M(X_1, \ldots, X_n)] = \frac{E[X_1] + \cdots + E[X_n]}{n} = \mu$$

The variance of $M(X_1, \ldots, X_n)$ is

$$\text{Var}[M(X_1, \ldots, X_n)] = \frac{\text{Var}[X_1 + \cdots + X_n]}{n^2} \leq \frac{\sigma^2(1+a)}{n(1-a)}$$

Applying the Chebyshev inequality to $M(X_1, \ldots, X_n)$ yields

$$P[|M(X_1, \ldots, X_n) - \mu| \geq c] \leq \frac{\text{Var}[M(X_1, \ldots, X_n)]}{c^2} \leq \frac{\sigma^2(1+a)}{n(1-a)c^2}$$

(c) Taking the limit as $n$ approaches infinity of the bound derived in part (b) yields

$$\lim_{n\to\infty} P[|M(X_1, \ldots, X_n) - \mu| \geq c] \leq \lim_{n\to\infty} \frac{\sigma^2(1+a)}{n(1-a)c^2} = 0$$

Thus

$$\lim_{n\to\infty} P[|M(X_1, \ldots, X_n) - \mu| \geq c] = 0$$

**Problem 8.4.3**

The solution to this problem generalizes the solution of Problem 8.4.1. As a preliminary step, first we find the CDF of $Y_n = \min\{X_1, \ldots, X_n\}$.

$$P[Y_n \leq y] = 1 - P[Y_n > y] = 1 - P[X_1 > y, X_2 > y, \cdots, X_n > y]$$

Since $X_1, X_2, \ldots$ is an iid sequence,

$$P[X_1 > y, X_2 > y, \cdots, X_n > y] = P[X_1 > y] \cdots P[X_n > y] = (1 - F_X(y))^n$$

This implies

$$P[Y_n \leq y] = 1 - (1 - F_X(y))^n$$

Now we use Theorem 8.7 to prove w.p.1 convergence. Since $F_{X_i}(0) = 0$, we know that each $X_i$ and thus each $Y_i$ is nonnegative. This implies

$$S_n(\varepsilon) = \{|Y_n| < \varepsilon\} = \{Y_n < \varepsilon\}$$

Since $Y_n = \min\{X_1, \ldots, X_n\}$, we observe that $Y_n = \min\{X_n, Y_{n-1}\}$. Hence $Y_n \leq Y_{n-1}$ for all $n$ so that if $Y_n \leq \varepsilon$, then $Y_k \leq \varepsilon$ for all $k \geq n$. Hence,

$$P[\cap_{k \geq n} S_k(\varepsilon)] = P[Y_n \leq \varepsilon, Y_{n+1} \leq \varepsilon, \ldots] = P[Y_n \leq \varepsilon] = 1 - (1 - F_X(\varepsilon))^n$$

Thus,

$$\lim_{n \to \infty} P[\cap_{k \geq n} S_k(\varepsilon)] = \lim_{n \to \infty} 1 - (1 - F_X(\varepsilon))^n = 1$$

Note that if there were an $\varepsilon_0$ such that $F_X(\varepsilon_0) = 0$, then

$$\lim_{n \to \infty} 1 - (1 - F_X(\varepsilon_0))^n = \lim_{n \to \infty} 1 - (1 - 0)^n = 0$$

and the $Y_n$ sequence would not converge to zero.

**Problem 8.4.4**

For $n = 1, 2, \ldots$, let $W_n = (X_{3n-2} + X_{3n-1} + X_{3n})/3$. That is, $W_1 = (X_1 + X_2 + X_3)/3$, $W_2 = (X_4 + X_5 + X_6)/3$ and so on. Note that $W_1, W_2, \ldots$ is an iid random sequence. In addition, since $E[X_{3n}] = 0$ for all $n$,

$$E[W_n] = \frac{E[X_{3n-2}] + E[X_{3n-1}] + E[X_{3n}]}{3} = 1/3$$

By the construction of $W_n$, we can write

$$Y_{3n} = \frac{X_1 + \cdots + X_{3n}}{3n} = \frac{W_1 + \cdots + W_n}{n}$$

Since $W_1, W_2, \ldots$ is an iid sequence with $E[W] = 1/3$, the strong law of large numbers says that

$$\lim_{k \to \infty} Y_{3k} = 1/3 \qquad \text{w.p. } 1$$

At this point it may seem that we are done. However, we have not quite shown that the sequence $Y_n$ converges w.p. 1. Given $Y_{3k}$, we need to find upper and lower bounds to $Y_j$. When $3k \leq j \leq 3k+2$,

$$Y_j = \frac{X_1 + \cdots + X_{3k}}{j} + \frac{X_{3k+1} + \cdots + X_j}{j}$$

Since $X_{3k+1} + \cdots + X_j \geq 0$,

$$Y_j \geq \frac{X_1 + \cdots + X_{3k}}{j} = \frac{3k}{j} Y_{3k} = \left(1 - \frac{j-3k}{j}\right) Y_{3k}$$

Since $0 \leq j - 3k \leq 2$ and since $Y_{3k} \leq 1$, we see that

$$Y_j \geq Y_{3k} - \frac{2}{j} Y_{3k} \geq Y_{3k} - \frac{2}{j}$$

In addition, since $j \leq 3k+2$,

$$X_{3k+1} + \cdots + X_j \leq X_{3k+1} + X_{3k+2} \leq 2$$

This implies

$$Y_j \leq \frac{3k}{j} \frac{X_1 + \cdots + X_{3k}}{3k} + \frac{2}{j} \leq Y_{3k} + \frac{2}{j}$$

Combining these results, we have for $3k \leq j \leq 3k+2$,

$$Y_{3k} - 2/j \leq Y_j \leq Y_{3k} + 2/j$$

We can rewrite this result using the floor function as

$$Y_{3\lfloor j/3 \rfloor} - 2/j \leq Y_j \leq Y_{3\lfloor j/3 \rfloor} + 2/j$$

Taking limits, we obtain with probability 1 that

$$\lim_{j \to \infty} Y_{3\lfloor j/3 \rfloor} - 2/j \leq \lim_{j \to \infty} Y_j \leq \lim_{j \to \infty} Y_{3\lfloor j/3 \rfloor} + 2/j$$

which implies

$$1/3 \leq \lim_{j \to \infty} Y_j \leq 1/3 \qquad \text{w.p. } 1$$