Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,6.2.4 6.3.4 6.4.2 6.4.3 6.5.6 6.5.7 6.6.3 6.7.1 6.7.6 6.7.8 6.8.5 6.9.6 and 6.9.7

Problem 6.2.4

The statement is *false*. As a counterexample, consider the rectified cosine waveform $X(t) = R|\cos 2\pi ft|$ of Example 6.8. When $t = \pi/2$, then $\cos 2\pi ft = 0$ so that $X(\pi/2) = 0$. Hence $X(\pi/2)$ has PDF

$$f_{X(\pi/2)}(x) = \delta(x)$$

That is, $X(\pi/2)$ is a discrete random variable.

Problem 6.3.4

Since the problem states that the pulse is delayed, we will assume $T \geq 0$. This problem is difficult because the answer will depend on t. In particular, for $t < 0$, $X(t) = 0$ and $f_{X(t)}(x) = \delta(x)$. Things are more complicated when $t > 0$. For $x < 0$, $P[X(t) > x] = 1$. For $x \geq 1$, $P[X(t) > x] = 0$. Lastly, for $0 \leq x < 1$,

$$P[X(t) > x] = P\left[e^{-(t-T)}u(t-T) > x\right] = P[t + \ln x < T \leq t] = F_T(t) - F_T(t + \ln x)$$

Note that condition $T \leq t$ is needed to make sure that the pulse doesn't arrive after time t. The other condition $T > t + \ln x$ ensures that the pulse didn't arrrive too early and already decay too much. We can express these facts in terms of the CDF of $X(t)$.

$$F_{X(t)}(x) = 1 - P[X(t) > x] = \begin{cases} 0 & x < 0 \\ 1 + F_T(t + \ln x) - F_T(t) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

We can take the derivative of the CDF to find the PDF. However, we need to keep in mind that the CDF has a jump discontinuity at $x = 0$. In particular, since $\ln 0 = -\infty$,

$$F_{X(t)}(0) = 1 + F_T(-\infty) - F_T(t) = 1 - F_T(t)$$

Hence, when we take a derivative, we will see an impulse at $x = 0$. The PDF of $X(t)$ is

$$f_{X(t)}(x) = \begin{cases} [1 - F_T(t)]\delta(x) + f_T(t + \ln x)/x & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 6.4.2

independent Gaussian random variables. Hence, each W_n must have the same PDF. That is, the W_n are identically distributed. However, since W_{n-1} and W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We can verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30$$

Next, we observe that W_{n-1} and W_n have covariance

$$\begin{aligned}\mathrm{Cov}[W_{n-1},W_n] &= E[W_{n-1}W_n] - E[W_n]E[W_{n-1}] \\ &= \frac{1}{4}E[(X_{n-1}+X_{n-2})(X_n+X_{n-1})] - 900\end{aligned}$$

We observe that for $n \neq m$, $E[X_nX_m] = E[X_n]E[X_m] = 900$ while

$$E\left[X_n^2\right] = \mathrm{Var}[X_n] + (E[X_n])^2 = 916$$

Thus,

$$\mathrm{Cov}[W_{n-1},W_n] = \frac{900+916+900+900}{4} - 900 = 4$$

Since $\mathrm{Cov}[W_{n-1},W_n] \neq 0$, W_n and W_{n-1} must be dependent.

Problem 6.4.3

successes $k-1$ and k is exactly $y \geq 0$ iff after success $k-1$, there are y failures followed by a success. Since the Bernoulli trials are independent, the probability of this event is $(1-p)^y p$. The complete PMF of Y_k is

$$P_{Y_k}(y) = \begin{cases} (1-p)^y p & y = 0,1,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Since this argument is valid for all k including $k=1$, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. Moreover, since the trials are independent, the failures between successes $k-1$ and k and the number of failures between successes $k'-1$ and k' are independent. Hence, $Y_1, Y_2, \ldots$ is an iid sequence.

Problem 6.5.6

The random variables K and J have PMFs

$$P_J(j) = \begin{cases} \frac{\alpha^j e^{-\alpha}}{j!} & j = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \qquad P_K(k) = \begin{cases} \frac{\beta^k e^{-\beta}}{k!} & k = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

For $n \geq 0$, we can find the PMF of $N = J + K$ via

$$P[N=n] = \sum_{k=-\infty}^{\infty} P[J=n-k, K=k]$$

Since J and K are independent, non-negative random variables,

$$\begin{aligned}P[N=n] &= \sum_{k=0}^{n} P_J(n-k)P_K(k) \\ &= \sum_{k=0}^{n} \frac{\alpha^{n-k}e^{-\alpha}}{(n-k)!}\frac{\beta^k e^{-\beta}}{k!} \\ &= \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} \underbrace{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\left(\frac{\alpha}{\alpha+\beta}\right)^{n-k}\left(\frac{\beta}{\alpha+\beta}\right)^{k}}_{1}\end{aligned}$$

The marked sum above equals 1 because it is the sum of a binomial PMF over all possible values. The PMF of N is the Poisson PMF

$$P_N(n) = \begin{cases} \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Problem 6.5.7

(a) For $X_i = -\ln U_i$, we can write

$$P[X_i > x] = P[-\ln U_i > x] = P[\ln U_i \leq -x] = P\left[U_i \leq e^{-x}\right]$$

When $x < 0$, $e^{-x} > 1$ so that $P[U_i \leq e^{-x}] = 1$. When $x \geq 0$, we have $0 < e^{-x} \leq 1$, implying $P[U_i \leq e^{-x}] = e^{-x}$. Combining these facts, we have

$$P[X_i > x] = \begin{cases} 1 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

This permits us to show that the CDF of X_i is

$$F_{X_i}(x) = 1 - P[X_i > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x > 0 \end{cases}$$

We see that X_i has an exponential CDF with mean 1.

(b) Note that $N = n$ iff

$$\prod_{i=1}^{n} U_i \geq e^{-t} > \prod_{i=1}^{n+1} U_i$$

By taking the logarithm of both inequalities, we see that $N = n$ iff

$$\sum_{i=1}^{n} \ln U_i \geq -t > \sum_{i=1}^{n+1} \ln U_i$$

Next, we multiply through by -1 and recall that $X_i = -\ln U_i$ is an exponential random variable. This yields $N = n$ iff

$$\sum_{i=1}^{n} X_i \leq t < \sum_{i=1}^{n+1} X_i$$

Now we recall that a Poisson process $N(t)$ of rate 1 has independent exponential interarrival times $X_1, X_2, \ldots$. That is, the ith arrival occurs at time $\sum_{j=1}^{i} X_j$. Moreover, $N(t) = n$ iff the first n arrivals occur by time t but arrival $n+1$ occurs after time t. Since the random variable $N(t)$ has a Poisson distribution with mean t, we can write

$$P\left[\sum_{i=1}^{n} X_i \leq t < \sum_{i=1}^{n+1} X_i\right] = P[N(t) = n] = \frac{t^n e^{-t}}{n!}$$

Problem 6.6.3
$Y_n = X_n - X_{n-1} = X(n) - X(n-1)$ is a Gaussian random variable with mean zero and variance α. Since this fact is true for all n, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. By Definition 6.11 for Brownian motion, $Y_n = X(n) - X(n-1)$ is independent of $X(m)$ for any $m \leq n-1$. Hence Y_n is independent of $Y_m = X(m) - X(m-1)$ for any $m \leq n-1$. Equivalently, $Y_1, Y_2, \ldots$ is a sequence of independent random variables.

Problem 6.7.1
The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)]$$

for $k = 0$, $C_X[m,0] = \text{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]E[(X_{m+k} - \mu_X)] = 0$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Problem 6.7.6
By repeated application of the recursion $C_n = C_{n-1}/2 + 4X_n$, we obtain

$$\begin{aligned} C_n &= \frac{C_{n-2}}{4} + 4\left[\frac{X_{n-1}}{2} + X_n\right] \\ &= \frac{C_{n-3}}{8} + 4\left[\frac{X_{n-2}}{4} + \frac{X_{n-1}}{2} + X_n\right] \\ &\vdots \\ &= \frac{C_0}{2^n} + 4\left[\frac{X_1}{2^{n-1}} + \frac{X_2}{2^{n-2}} + \cdots + X_n\right] \\ &= \frac{C_0}{2^n} + 4\sum_{i=1}^{n} \frac{X_i}{2^{n-i}} \end{aligned}$$

(a) Since $C_0, X_1, X_2, \ldots$ all have zero mean,

$$E[C_n] = \frac{E[C_0]}{2^n} + 4\sum_{i=1}^{n} \frac{E[X_i]}{2^{n-i}} = 0$$

(b) The autocovariance is

$$C_C[m,k] = E\left[\left(\frac{C_0}{2^n} + 4\sum_{i=1}^{n} \frac{X_i}{2^{n-i}}\right)\left(\frac{C_0}{2^m + k} + 4\sum_{j=1}^{m+k} \frac{X_j}{2^{m+k-j}}\right)\right]$$

Since $C_0, X_1, X_2, \ldots$ are independent (and zero mean), $E[C_0 X_i] = 0$. This implies

$$C_C[m,k] = \frac{E\left[C_0^2\right]}{2^{2m+k}} + 16\sum_{i=1}^{m}\sum_{j=1}^{m+k} \frac{E\left[X_i X_j\right]}{2^{m-i}2^{m+k-j}}$$

For $i \neq j$, $E\left[X_i X_j\right] = 0$ so that only the $i = j$ terms make any contribution to the double sum. However, at this point, we must consider the cases $k \geq 0$ and $k < 0$ separately. Since each X_i has variance 1, the autocovariance for $k \geq 0$ is

$$\begin{aligned} C_C[m,k] &= \frac{1}{2^{2m+k}} + 16 \sum_{i=1}^{m} \frac{1}{2^{2m+k-2i}} \\ &= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \sum_{i=1}^{m} (1/4)^{m-i} \\ &= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \frac{1-(1/4)^m}{3/4} \end{aligned}$$

For $k < 0$, we can write

$$\begin{aligned} C_C[m,k] &= \frac{E\left[C_0^2\right]}{2^{2m+k}} + 16 \sum_{i=1}^{m} \sum_{j=1}^{m+k} \frac{E\left[X_i X_j\right]}{2^{m-i} 2^{m+k-j}} \\ &= \frac{1}{2^{2m+k}} + 16 \sum_{i=1}^{m+k} \frac{1}{2^{2m+k-2i}} \\ &= \frac{1}{2^{2m+k}} + \frac{16}{2^{-k}} \sum_{i=1}^{m+k} (1/4)^{m+k-i} \\ &= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \frac{1-(1/4)^{m+k}}{3/4} \end{aligned}$$

A general expression that's valid for all m and k is

$$C_C[m,k] = \frac{1}{2^{2m+k}} + \frac{16}{2^{|k|}} \frac{1-(1/4)^{\min(m,m+k)}}{3/4}$$

(c) Since $E[C_i] = 0$ for all i, our model has a mean daily temperature of zero degrees Celsius for the entire year. This is not a reasonable model for a year.

(d) For the month of January, a mean temperature of zero degrees Celsius seems quite reasonable. we can calculate the variance of C_n by evaluating the covariance at $n = m$. This yields

$$\operatorname{Var}[C_n] = \frac{1}{4^n} + \frac{16}{4^n} \frac{4(4^n - 1)}{3}$$

Note that the variance is upper bounded by

$$\operatorname{Var}[C_n] \leq 64/3$$

Hence the daily temperature has a standard deviation of $8/\sqrt{3} \approx 4.6$ degrees. Without actual evidence of daily temperatures in January, this model is more difficult to discredit.

Problem 6.7.8
process covariance is almost identical to the derivation of the Brownian motion autocovariance since both rely on the use of independent increments. From the definition of the Poisson process, we know that $\mu_N(t) = \lambda t$. When $s < t$, we can write

$$\begin{aligned} C_N(s,t) &= E[N(s)N(t)] - (\lambda s)(\lambda t) \\ &= E[N(s)[(N(t) - N(s)) + N(s)]] - \lambda^2 st \\ &= E[N(s)[N(t) - N(s)]] + E\left[N^2(s)\right] - \lambda^2 st \end{aligned}$$

By the definition of the Poisson process, $N(s)$ and $N(t) - N(s)$ are independent for $s < t$. This implies

$$E[N(s)[N(t) - N(s)]] = E[N(s)]E[N(t) - N(s)] = \lambda s(\lambda t - \lambda s)$$

Note that since $N(s)$ is a Poisson random variable, $\text{Var}[N(s)] = \lambda s$. Hence

$$E\left[N^2(s)\right] = \text{Var}[N(s)] + (E[N(s)]^2 = \lambda s + (\lambda s)^2$$

Therefore, for $s < t$,

$$C_N(s,t) = \lambda s(\lambda t - \lambda s) + \lambda s + (\lambda s)^2 - \lambda^2 st = \lambda s$$

If $s > t$, then we can interchange the labels s and t in the above steps to show $C_N(s,t) = \lambda t$. For arbitrary s and t, we can combine these facts to write

$$C_N(s,t) = \lambda \min(s,t)$$

Problem 6.8.5

Since $g(\cdot)$ is an unspecified function, we will work with the joint CDF of $Y(t_1+\tau), \ldots, Y(t_n+\tau)$. To show $Y(t)$ is a stationary process, we will show that for all τ,

$$F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1, \ldots, y_n) = F_{Y(t_1),\ldots,Y(t_n)}(y_1, \ldots, y_n)$$

By taking partial derivatives with respect to $y_1, \ldots, y_n$, it should be apparent that this implies that the joint PDF $f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1, \ldots, y_n)$ will not depend on τ. To proceed, we write

$$\begin{aligned} F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1, \ldots, y_n) &= P[Y(t_1+\tau) \le y_1, \ldots, Y(t_n+\tau) \le y_n] \\ &= P\Bigg[\underbrace{g(X(t_1+\tau)) \le y_1, \ldots, g(X(t_n+\tau)) \le y_n}_{A_\tau}\Bigg] \end{aligned}$$

In principle, we can calculate $P[A_\tau]$ by integrating $f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1, \ldots, x_n)$ over the region corresponding to event A_τ. Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1, \ldots, x_n) = f_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n)$$

This implies $P[A_\tau]$ does not depend on τ. In particular,

$$\begin{aligned} F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1, \ldots, y_n) &= P[A_\tau] \\ &= P[g(X(t_1)) \le y_1, \ldots, g(X(t_n)) \le y_n] \\ &= F_{Y(t_1),\ldots,Y(t_n)}(y_1, \ldots, y_n) \end{aligned}$$

Problem 6.9.6

with this problem since X_n is not defined for $n<0$. This implies $C_X[n,k]$ is not defined for $k<-n$ and thus $C_X[n,k]$ cannot be completely independent of k. When n is large, corresponding to a process that has been running for a long time, this is a technical issue, and not a practical concern. Instead, we will find $\bar{\sigma}^2$ such that $C_X[n,k]=C_X[k]$ for all n and k for which the covariance function is defined. To do so, we need to express X_n in terms of $Z_0, Z_1, \ldots, Z_{n_1}$. We do this in the following way:

$$
\begin{aligned}
X_n &= cX_{n-1}+Z_{n-1} \\
&= c[cX_{n-2}+Z_{n-2}]+Z_{n-1} \\
&= c^2[cX_{n-3}+Z_{n-3}]+cZ_{n-2}+Z_{n-1} \\
&\vdots \\
&= c^n X_0 + c^{n-1}Z_0 + c^{n-2}Z_2+\cdots+Z_{n-1} \\
&= c^n X_0 + \sum_{i=0}^{n-1} c^{n-1-i} Z_i
\end{aligned}
$$

Since $E[Z_i]=0$, the mean function of the X_n process is

$$E[X_n]=c^n E[X_0]+\sum_{i=0}^{n-1} c^{n-1-i}E[Z_i]=E[X_0]$$

Thus, for X_n to be a zero mean process, we require that $E[X_0]=0$. The autocorrelation function can be written as

$$R_X[n,k]=E[X_nX_{n+k}]=E\left[\left(c^nX_0+\sum_{i=0}^{n-1}c^{n-1-i}Z_i\right)\left(c^{n+k}X_0+\sum_{j=0}^{n+k-1}c^{n+k-1-j}Z_j\right)\right]$$

Although it was unstated in the problem, we will assume that X_0 is independent of $Z_0, Z_1, \ldots$ so that $E[X_0Z_i]=0$. Since $E[Z_i]=0$ and $E\left[Z_iZ_j\right]=0$ for $i\neq j$, most of the cross terms will drop out. For $k\geq 0$, autocorrelation simplifies to

$$R_X[n,k]=c^{2n+k}\operatorname{Var}[X_0]+\sum_{i=0}^{n-1}c^{2(n-1)+k-2i}\bar{\sigma}^2=c^{2n+k}\operatorname{Var}[X_0]+\bar{\sigma}^2c^k\frac{1-c^{2n}}{1-c^2}$$

Since $E[X_n]=0$, $\operatorname{Var}[X_0]=R_X[n,0]=\sigma^2$ and we can write for $k\geq 0$,

$$R_X[n,k]=\bar{\sigma}^2\frac{c^k}{1-c^2}+c^{2n+k}\left(\sigma^2-\frac{\bar{\sigma}^2}{1-c^2}\right)$$

For $k < 0$, we have

$$
\begin{aligned}
R_X[n,k] &= E\left[\left(c^n X_0 + \sum_{i=0}^{n-1} c^{n-1-i} Z_i\right)\left(c^{n+k} X_0 + \sum_{j=0}^{n+k-1} c^{n+k-1-j} Z_j\right)\right] \\
&= c^{2n+k}\operatorname{Var}[X_0] + c^{-k} \sum_{j=0}^{n+k-1} c^{2(n+k-1-j)} \bar{\sigma}^2 \\
&= c^{2n+k}\sigma^2 + \bar{\sigma}^2 c^{-k} \frac{1-c^{2(n+k)}}{1-c^2} \\
&= \frac{\bar{\sigma}^2}{1-c^2} c^{-k} + c^{2n+k}\left(\sigma^2 - \frac{\bar{\sigma}^2}{1-c^2}\right)
\end{aligned}
$$

We see that $R_X[n,k] = \sigma^2 c^{|k|}$ by choosing

$$\bar{\sigma}^2 = (1-c^2)\sigma^2$$

Problem 6.9.7

We can recusively solve for Y_n as follows.

$$
\begin{aligned}
Y_n &= aX_n + aY_{n-1} \\
&= aX_n + a[aX_{n-1} + aY_{n-2}] \\
&= aX_n + a^2 X_{n-1} + a^2[aX_{n-2} + aY_{n-3}]
\end{aligned}
$$

By continuing the same procedure, we can conclude that

$$Y_n = \sum_{j=0}^{n} a^{j+1} X_{n-j} + a^n Y_0$$

Since $Y_0 = 0$, the substitution $i = n - j$ yields

$$Y_n = \sum_{i=0}^{n} a^{n-i+1} X_i$$

Now we can calculate the mean

$$E[Y_n] = E\left[\sum_{i=0}^{n} a^{n-i+1} X_i\right] = \sum_{i=0}^{n} a^{n-i+1} E[X_i] = 0$$

To calculate the autocorrelation $R_Y[m,k]$, we consider first the case when $k \geq 0$.

$$C_Y[m,k] = E\left[\sum_{i=0}^{m} a^{m-i+1} X_i \sum_{j=0}^{m+k} a^{m+k-j+1} X_j\right] = \sum_{i=0}^{m}\sum_{j=0}^{m+k} a^{m-i+1} a^{m+k-j+1} E\left[X_i X_j\right]$$

Since the X_i is a sequence of iid standard normal random variables,

$$E\left[X_i X_j\right] = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus, only the $i = j$ terms make a nonzero contribution. This implies

$$\begin{aligned}
C_Y[m,k] &= \sum_{i=0}^{m} a^{m-i+1} a^{m+k-i+1} \\
&= a^k \sum_{i=0}^{m} a^{2(m-i+1)} \\
&= a^k \left[(a^2)^{m+1} + (a^2)^m + \cdots + a^2 \right] \\
&= \frac{a^2}{1-a^2} a^k \left[1 - (a^2)^{m+1} \right]
\end{aligned}$$

For $k \leq 0$, we start from

$$C_Y[m,k] = \sum_{i=0}^{m} \sum_{j=0}^{m+k} a^{m-i+1} a^{m+k-j+1} E\left[X_i X_j\right]$$

As in the case of $k \geq 0$, only the $i = j$ terms make a contribution. Also, since $m+k \leq m$,

$$C_Y[m,k] == \sum_{j=0}^{m+k} a^{m-j+1} a^{m+k-j+1} = a^{-k} \sum_{j=0}^{m+k} a^{m+k-j+1} a^{m+k-j+1}$$

By steps quite similar to those for $k \geq 0$, we can show that

$$C_Y[m,k] = \frac{a^2}{1-a^2} a^{-k} \left[1 - (a^2)^{m+k+1} \right]$$

A general expression that is valid for all m and k would be

$$C_Y[m,k] = \frac{a^2}{1-a^2} a^{|k|} \left[1 - (a^2)^{\min(m,m+k)+1} \right]$$

Since $C_Y[m,k]$ depends on m, the Y_n process is not wide sense stationary.