Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Roy D. Yates and David J. Goodman

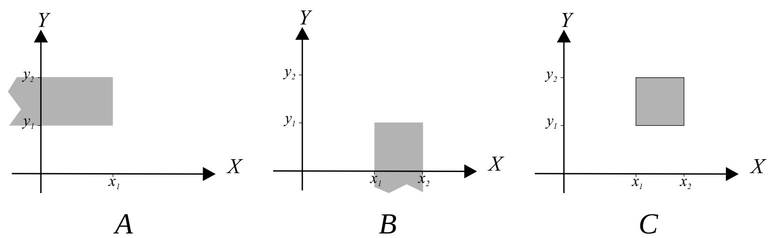
Problem Solutions : Yates and Goodman,5.1.5 5.1.6 5.2.4 5.3.6 5.4.5 5.5.4 5.6.4 5.7.5 5.7.6 5.8.4 5.8.6 5.8.7 5.8.8 5.9.5 and 5.10.4

Problem 5.1.5

Theorem 5.3 which states

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

(a) The events A, B, and C are



(b) In terms of the joint CDF $F_{X,Y}(x,y)$, we can write

$$\begin{aligned} P[A] &= F_{X,Y}(x_1,y_2) - F_{X,Y}(x_1,y_1) \\ P[B] &= F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_1) \\ P[A \cup B \cup C] &= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_1) \end{aligned}$$

(c) Since A, B, and C are mutually exclusive,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C]$$

However, since we want to express $P[C] = P[x_1 < X \leq x_2, y_1 < Y \leq y_2]$ in terms of the joint CDF $F_{X,Y}(x,y)$, we write

$$\begin{aligned} P[C] &= P[A \cup B \cup C] - P[A] - P[B] \\ &= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1) \end{aligned}$$

which completes the proof of the theorem.

Problem 5.1.6

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-(x+y)} & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

First, we find the CDF $F_X(x)$ and $F_Y(y)$.

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1 & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence, for any $x \geq 0$ or $y \geq 0$,

$$P[X > x] = 0 \qquad P[Y > y] = 0$$

For $x \geq 0$ and $y \geq 0$, this implies

$$P[\{X > x\} \cup \{Y > y\}] \leq P[X > x] + P[Y > y] = 0$$

However,

$$P[\{X > x\} \cup \{Y > y\}] = 1 - P[X \leq x, Y \leq y] = 1 - (1 - e^{-(x+y)}) = e^{-(x+y)}$$

Thus, we have the contradiction that $e^{-(x+y)} \leq 0$ for all $x, y \geq 0$. We can conclude that the given function is not a valid CDF.

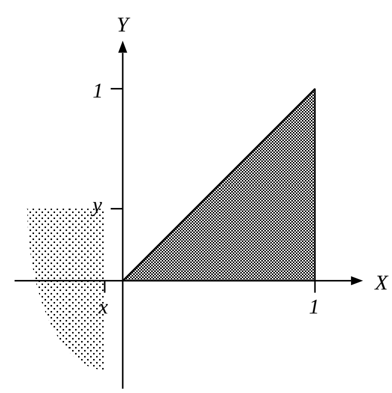
Problem 5.2.4

The only difference between this problem and Example 5.2 is that in this problem we must integrate the joint PDF over the regions to find the probabilities. Just as in Example 5.2, there are five cases. We will use variable u and v as dummy variables for x and y.

- $x < 0$ or $y < 0$

In this case, the region of integration doesn't overlap the region of nonzero probability and

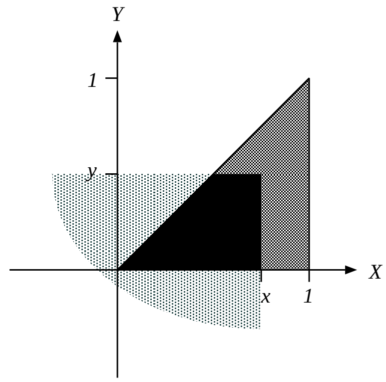
$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv = 0$$



- $0 < y \leq x \leq 1$

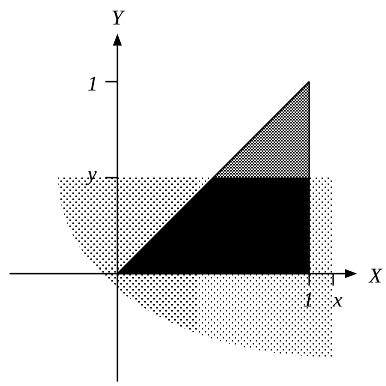
In this case, the region where the integral has a nonzero contribution is

$$
\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\, dy\, dx \\
&= \int_{0}^{y}\int_{v}^{x} 8uv\, du\, dv \\
&= \int_{0}^{y} 4(x^2 - v^2) v\, dv \\
&= 2x^2v^2 - v^4\Big|_{v=0}^{v=y} = 2x^2y^2 - y^4
\end{aligned}
$$



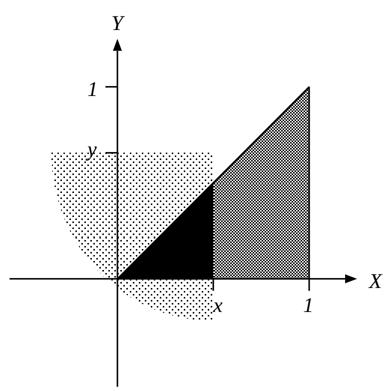
- $0 < x \leq y$ and $0 \leq x \leq 1$

$$
\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\, dv\, du \\
&= \int_{0}^{x}\int_{0}^{u} 8uv\, dv\, du \\
&= \int_{0}^{x} 4u^3\, du \\
&= x^4
\end{aligned}
$$



- $0 < y \leq 1$ and $x \geq 1$

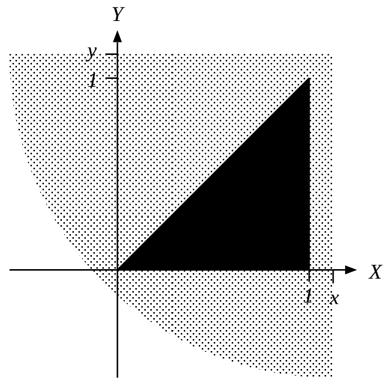
$$
\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\, dv\, du \\
&= \int_{0}^{y}\int_{v}^{1} 8uv\, du\, dv \\
&= \int_{0}^{y} 4v(1 - v^2)\, dv \\
&= 2y^2 - y^4
\end{aligned}
$$



- $x \geq 1$ and $y \geq 1$

In this case, the region of integration completely covers the region of nonzero probability and

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv = 1$$



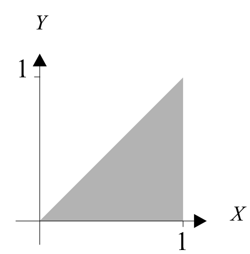
The complete answer for the joint CDF is

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ 2x^2y^2 - y^4 & 0 < y \leq x \leq 1 \\ x^4 & 0 \leq x \leq y, 0 \leq x \leq 1 \\ 2y^2 - y^4 & 0 \leq y \leq 1, x \geq 1 \\ 1 & x \geq 1, y \geq 1 \end{cases}$$

Problem 5.3.6

(a) The joint PDF of X and Y and the region of nonzero probability are

$$f_{X,Y}(x,y) = \begin{cases} cy & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



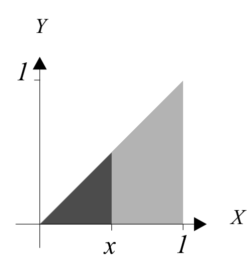
(b) To find the value of the constant, c, we integrate the joint PDF over all x and y.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = \int_0^1 \int_0^x cy\, dy\, dx = \int_0^1 \frac{cx^2}{2}\, dx = \left.\frac{cx^3}{6}\right|_0^1 = \frac{c}{6}$$

Thus $c = 6$.

(c) We can find the CDF $F_X(x) = P[X \leq x]$ by integrating the joint PDF over the event $X \leq x$. For $x < 0$, $F_X(x) = 0$. For $x > 1$, $F_X(x) = 1$. For $0 \leq x \leq 1$,

$$\begin{aligned} F_X(x) &= \iint_{x' \leq x} f_{X,Y}(x',y')\, dy'\, dx' \\ &= \int_0^x \int_0^{x'} 6y'\, dy'\, dx' \\ &= \int_0^x 3(x')^2\, dx' = x^3 \end{aligned}$$

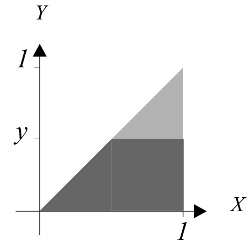


The complete expression for the joint CDF is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

(d) Similarly, we find the CDF of Y by integrating $f_{X,Y}(x,y)$ over the event $Y \leq y$. For $y < 0$, $F_Y(y) = 0$ and for $y > 1$, $F_Y(y) = 1$. For $0 \leq y \leq 1$,

$$\begin{aligned} F_Y(y) &= \iint_{y' \leq y} f_{X,Y}(x', y')\, dy'\, dx' \\ &= \int_0^y \int_{y'}^1 6y'\, dx'\, dy' \\ &= \int_0^y 6y'(1-y')\, dy' = 3(y')^2 - 2(y')^3\Big|_0^y = 3y^2 - 2y^3 \end{aligned}$$

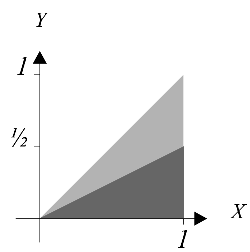


The complete expression for the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

(e) To find $P[Y \leq X/2]$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region $y \leq x/2$.

$$\begin{aligned} P[Y \leq X/2] &= \int_0^1 \int_0^{x/2} 6y\, dy\, dx \\ &= \int_0^1 3y^2\Big|_0^{x/2} dx \\ &= \int_0^1 \frac{3x^2}{4}\, dx = 1/4 \end{aligned}$$



Problem 5.4.5

The position of the mobile phone is equally likely to be anywhere in the area of a circle with radius 16 km. Let X and Y denote the position of the mobile. Since we are given that the cell has a radius of 4 km, we will measure X and Y in kilometers. Assuming the base station is at the origin of the X,Y plane, the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{16\pi} & x^2 + y^2 \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

Since the radial distance of the mobile from the base station is $R = \sqrt{X^2 + Y^2}$, the CDF of R is

$$F_R(r) = P[R \leq r] = P\left[X^2 + Y^2 \leq r\right]$$

By changing to polar coordinates, we see that for $0 \leq r \leq 4$ km,

$$F_R(r) = \int_0^{2\pi} \int_0^r \frac{r'}{16\pi}\, dr'\, d\theta' = r^2/16$$

So

$$F_R(r) = \begin{cases} 0 & r < 0 \\ r^2/16 & 0 \le r < 4 \\ 1 & r \ge 4 \end{cases}$$

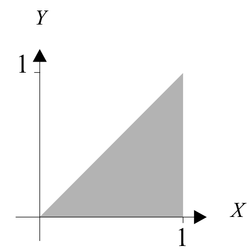
Then by taking the derivative with respect to r we arrive at the PDF

$$f_R(r) = \begin{cases} r/8 & 0 \le r \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.5.4

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Before finding moments, it is helpful to first find the marginal PDFs. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^x 2\, dy = 2x$$

Note that $f_X(x) = 0$ for $x < 0$ or $x > 1$. For $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^1 2\, dx = 2(1-y)$$

Also, for $y < 0$ or $y > 1$, $f_Y(y) = 0$. Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) The first two moments of X are

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx = 2/3$$

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 2x^3\, dx = 1/2$$

The variance of X is $\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 1/2 - 4/9 = 1/18$.

(b) The expected value and second moment of Y are

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 2y(1-y)\, dy = y^2 - \left.\frac{2y^3}{3}\right|_0^1 = 1/3$$

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^1 2y^2(1-y)\, dy = \left.\frac{2y^3}{3} - \frac{y^4}{2}\right|_0^1 = 1/6$$

The variance of Y is $\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 1/6 - 1/9 = 1/18$.

(c) Before finding the covariance, we find the correlation

$$E[XY] = \int_0^1 \int_0^x 2xy\,dy\,dx = \int_0^1 x^3\,dx = 1/4$$

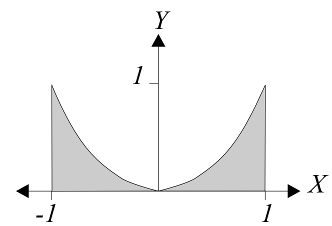
The covariance is $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 1/36$.

(d) $E[X+Y] = E[X] + E[Y] = 2/3 + 1/3 = 1$

(e) By Theorem 5.10, $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 1/6$.

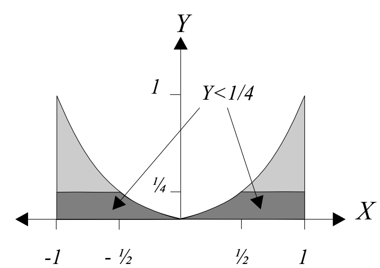
Problem 5.6.4

$$f_{X,Y}(x,y) = \begin{cases} \frac{5x^2}{2} & -1 \le x \le 1, 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases}$$



(a) The event $A = \{Y \le 1/4\}$ has probability

$$\begin{aligned} P[A] &= 2\int_0^{1/2}\int_0^{x^2} \frac{5x^2}{2}\,dy\,dx + 2\int_{1/2}^1 \int_0^{1/4} \frac{5x^2}{2}\,dy\,dx \\ &= \int_0^{1/2} 5x^4\,dx + \int_{1/2}^1 \frac{5x^2}{4}\,dx \\ &= x^5\Big|_0^{1/2} + 5x^3/12\Big|_{1/2}^1 = 19/48 \end{aligned}$$



This implies

$$\begin{aligned} f_{X,Y|A}(x,y) &= \begin{cases} f_{X,Y}(x,y)/P[A] & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 120x^2/19 & -1 \le x \le 1, 0 \le y \le x^2, y \le 1/4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} f_{Y|A}(y) &= \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dx = 2\int_{\sqrt{y}}^1 \frac{120x^2}{19}\,dx \\ &= \begin{cases} \frac{80}{19}(1 - y^{3/2}) & 0 \le y \le 1/4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c) The conditional expectation of Y given A is

$$E[Y|A] = \int_0^{1/4} y\frac{80}{19}(1 - y^{3/2})\,dy = \frac{80}{19}\left(\frac{y^2}{2} - \frac{2y^{7/2}}{7}\right)\Bigg|_0^{1/4} = \frac{65}{532}$$

(d) To find $f_{X|A}(x)$, we can write

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dy$$

However, when we substitute $f_{X,Y|A}(x,y)$, the limits will depend on the value of x. When $|x| \leq 1/2$, we have

$$f_{X|A}(x) = \int_0^{x^2} \frac{120x^2}{19}\, dy = \frac{120x^4}{19}$$

When $-1 \leq x \leq -1/2$ or $1/2 \leq x \leq 1$,

$$f_{X|A}(x) = \int_0^{1/4} \frac{120x^2}{19}\, dy = \frac{30x^2}{19}$$

The complete expression for the conditional PDF of X given A is

$$f_{X|A}(x) = \begin{cases} 30x^2/19 & -1 \leq x \leq -1/2 \\ 120x^4/19 & -1/2 \leq x \leq 1/2 \\ 30x^2/19 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

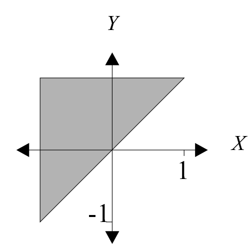
(e) The conditional mean of X given A is

$$E[X|A] = \int_{-1}^{-1/2} \frac{30x^3}{19}\, dx + \int_{-1/2}^{1/2} \frac{120x^5}{19}\, dx + \int_{1/2}^{1} \frac{30x^3}{19}\, dx = 0$$

Problem 5.7.5

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



(a) For $-1 \leq y \leq 1$, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \frac{1}{2}\int_{-1}^{y} dx = (y+1)/2$$

The complete expression for the marginal PDF of Y is

$$f_Y(y) = \begin{cases} (y+1)/2 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) The conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1+y} & -1 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(c) Given $Y = y$, the conditional PDF of X is uniform over $[-1, y]$. Hence the conditional expected value is $E[X|Y=y] = (y-1)/2$.

Problem 5.7.6
the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & 0 \le x^2 + y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) The marginal PDF of X is

$$f_X(x) = 2\int_0^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \le x \le r \\ 0 & \text{otherwise} \end{cases}$$

The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/(2\sqrt{r^2-x^2}) & y^2 \le r^2 - x^2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Given $X = x$, we observe that over the interval $[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}]$, Y has a uniform PDF. Since the conditional PDF $f_{Y|X}(y|x)$ is symmetric about $y = 0$,

$$E[Y|X = x] = 0$$

Problem 5.8.4
X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For the event $A = \{X > Y\}$, this problem asks us to calculate the conditional expectations $E[X|A]$ and $E[Y|A]$. We will do this using the conditional joint PDF $f_{X,Y|A}(x,y)$. Since X and Y are independent, it is tempting to argue that the event $X > Y$ does not alter the probability model for X and Y. Unfortunately, this is not the case. When we learn that $X > Y$, it increases the probability that X is large and Y is small. We will see this when we compare the conditional expectations $E[X|A]$ and $E[Y|A]$ to $E[X]$ and $E[Y]$.

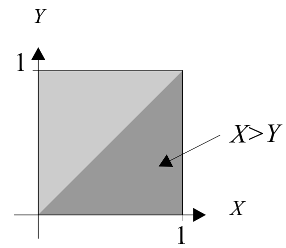
(a) We can calculate the unconditional expectations, $E[X]$ and $E[Y]$, using the marginal PDFs $f_X(x)$ and $f_Y(y)$.

$$E[X] = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 2x^2\, dx = 2/3$$
$$E[Y] = \int_{-\infty}^{\infty} f_Y(y)\, dy = \int_0^1 3y^3\, dy = 3/4$$

(b) First, we need to calculate the conditional joint PDF $f_{X,Y|A}(x,y|a)\, x, y$. The first step is to write down the joint PDF of X and Y:

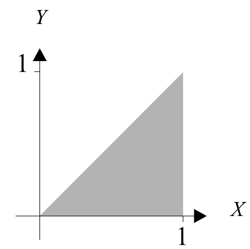
$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 6xy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The event A has probability



$$
\begin{aligned}
P[A] &= \iint_{x>y} f_{X,Y}(x,y)\,dy\,dx \\
&= \int_0^1 \int_0^x 6xy^2\,dy\,dx \\
&= \int_0^1 2x^4\,dx = 2/5
\end{aligned}
$$

The conditional joint PDF of X and Y given A is



$$
\begin{aligned}
f_{X,Y|A}(x,y) &= \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 15xy^2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

The triangular region of nonzero probability is a signal that given A, X and Y are no longer independent. The conditional expected value of X given A is

$$
\begin{aligned}
E[X|A] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X,Y|A}(x,y|a)\, x,y\,dy\,dx \\
&= 15 \int_0^1 x^2 \int_0^x y^2\,dy\,dx \\
&= 5 \int_0^1 x^5\,dx = 5/6
\end{aligned}
$$

The conditional expected value of Y given A is

$$
E[Y|A] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{X,Y|A}(x,y)\,dy\,dx = 15 \int_0^1 x \int_0^x y^3\,dy\,dx = \frac{15}{4} \int_0^1 x^5\,dx = 5/8
$$

We see that $E[X|A] > E[X]$ while $E[Y|A] < E[Y]$. That is, learning $X > Y$ gives us a clue that X may be larger than usual while Y may be smaller than usual.

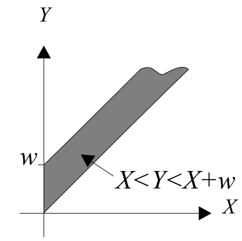
Problem 5.8.6

Random variables X and Y have joint PDF

$$
f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}
$$

For $W = Y - X$ we can find $f_W(w)$ by integrating over the region indicated in the figure below to get $F_W(w)$ then taking the derivative with respect to w. Since $Y \ge X$, $W = Y - X$ is nonnegative. Hence $F_W(w) = 0$ for $w < 0$. For $w \ge 0$,

$$
\begin{aligned}
F_W(w) &= 1 - P[W > w] = 1 - P[Y > X + w] \\
&= 1 - \int_0^\infty \int_{x+w}^\infty \lambda^2 e^{-\lambda y}\, dy\, dx \\
&= 1 - e^{-\lambda w}
\end{aligned}
$$



The complete expressions for the joint CDF and corresponding joint PDF are

$$
F_W(w) = \begin{cases} 0 & w < 0 \\ 1 - e^{-\lambda w} & w \geq 0 \end{cases} \qquad f_W(w) = \begin{cases} 0 & w < 0 \\ \lambda e^{-\lambda w} & w \geq 0 \end{cases}
$$

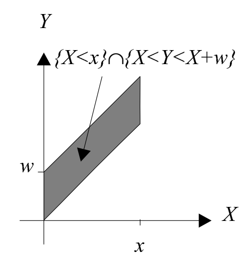
Problem 5.8.7

(a) To find if W and X are independent, we must be able to factor the joint density function $f_{X,W}(x,w)$ into the product $f_X(x)\, f_W(w)$ of marginal density functions. To verify this, we must find the joint PDF of X and W. First we find the joint CDF.

$$
F_{X,W}(x,w) = P[X \leq x, W \leq w] = P[X \leq x, Y - X \leq w] = P[X \leq x, Y \leq X + w]
$$

Since $Y \geq X$, the CDF of W satisfies $F_{X,W}(x,w) = P[X \leq x, X \leq Y \leq X + w]$. Thus, for $x \geq 0$ and $w \geq 0$,

$$
\begin{aligned}
F_{X,W}(x,w) &= \int_0^x \int_{x'}^{x'+w} \lambda^2 e^{-\lambda y}\, dy\, dx' \\
&= \int_0^x \left(-\lambda e^{-\lambda y} \Big|_{x'}^{x'+w} \right) dx' \\
&= \int_0^x \left(-\lambda e^{-\lambda(x'+w)} + \lambda e^{-\lambda x'} \right) dx' \\
&= e^{-\lambda(x'+w)} - e^{-\lambda x'} \Big|_0^x \\
&= (1 - e^{-\lambda x})(1 - e^{-\lambda w})
\end{aligned}
$$



We see that $F_{X,W}(x,w) = F_X(x) F_W(w)$. Moreover, by applying Theorem 5.2,

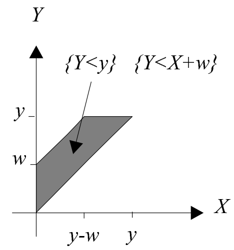
$$
f_{X,W}(x,w) = \frac{\partial^2 F_{X,W}(x,w)}{\partial x \partial w} = \lambda e^{-\lambda x} \lambda e^{-\lambda w} = f_X(x)\, f_W(w)
$$

Since we have our desired factorization, W and X are independent.

(b) Following the same procedure, we find the joint CDF of Y and W.

$$
F_{W,Y}(w,y) = P[W \leq w, Y \leq y] = P[Y - X \leq w, Y \leq y] = P[Y \leq X + w, Y \leq y]
$$

The region of integration corresponding to the event $\{Y \leq x + w, Y \leq y\}$ depends on whether $y < w$ or $y \geq w$. Keep in mind that although $W = Y - X \leq Y$, the dummy arguments y and w of $f_{W,Y}(w,y)$ need not obey the same constraints. In any case, we must consider each case separately. For $y > w$, the region of integration resembles

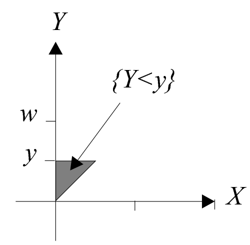


Thus for $y > w$, the integration is

$$\begin{aligned}
F_{W,Y}(w,y) &= \int_0^{y-w}\int_u^{u+w} \lambda^2 e^{-\lambda v}\,dv\,du + \int_{y-w}^{y}\int_u^{y} \lambda^2 e^{-\lambda v}\,dv\,du \\
&= \lambda\int_0^{y-w}\left[e^{-\lambda u} - e^{-\lambda(u+w)}\right]du + \lambda\int_{y-w}^{y}\left[e^{-\lambda u} - e^{-\lambda y}\right]du \\
&= \left[-e^{-\lambda u} + e^{-\lambda(u+w)}\right]\Big|_0^{y-w} + \left[-e^{-\lambda u} - u\lambda e^{-\lambda y}\right]\Big|_{y-w}^{y} \\
&= 1 - e^{-\lambda w} - \lambda w e^{-\lambda y}
\end{aligned}$$

For $y \le w$,

$$\begin{aligned}
F_{W,Y}(w,y) &= \int_0^{y}\int_u^{y} \lambda^2 e^{-\lambda v}\,dv\,du \\
&= \int_0^{y}\left[-\lambda e^{-\lambda y} + \lambda e^{-\lambda u}\right]du \\
&= -\lambda u e^{-\lambda y} - e^{-\lambda u}\Big|_0^{y} \\
&= 1 - (1+\lambda y)e^{-\lambda y}
\end{aligned}$$



The complete expression for the joint CDF is

$$F_{W,Y}(w,y) = \begin{cases} 1 - e^{-\lambda w} - \lambda w e^{-\lambda y} & 0 \le w \le y \\ 1 - (1+\lambda y)e^{-\lambda y} & 0 \le y \le w \\ 0 & \text{otherwise} \end{cases}$$

Applying Theorem 5.2 yields

$$f_{W,Y}(w,y) = \frac{\partial^2 F_{W,Y}(w,y)}{\partial w\,\partial y} = \begin{cases} 2\lambda^2 e^{-\lambda y} & 0 \le w \le y \\ 0 & \text{otherwise} \end{cases}$$

The joint PDF $f_{W,Y}(w,y)$ doesn't factor and thus W and Y are dependent.

Problem 5.8.8

We need to define the events $A = \{U \le u\}$ and $B = \{V \le v\}$. In this case,

$$F_{U,V}(u,v) = P[AB] = P[B] - P[A^c B] = P[V \le v] - P[U > u, V \le v]$$

Note that $U = \min(X,Y) > u$ if and only if $X > u$ and $Y > u$. In the same way, since $V = \max(X,Y)$, $V \le v$ if and only if $X \le v$ and $Y \le v$. Thus

$$P[U > u, V \le v] = P[X > u, Y > u, X \le v, Y \le v] = P[u < X \le v, u < Y \le v]$$

Thus, the joint CDF of U and V satisfies

$$F_{U,V}(u,v) = P[V \le v] - P[U > u, V \le v] = P[X \le v, Y \le v] - P[u < X \le v, u < X \le v]$$

Since X and Y are independent random variables,

$$\begin{aligned} F_{U,V}(u,v) &= P[X \le v]P[Y \le v] - P[u < X \le v]P[u < X \le v] \\ &= F_X(v)F_Y(v) - (F_X(v) - F_X(u))(F_Y(v) - F_Y(u)) \\ &= F_X(v)F_Y(u) + F_X(u)F_Y(v) - F_X(u)F_Y(u) \end{aligned}$$

The joint PDF is

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\partial^2 F_{U,V}(u,v)}{\partial u \partial v} \\ &= \frac{\partial}{\partial u}[f_X(v)F_Y(u) + F_X(u)f_Y(v)] \\ &= f_X(u)f_Y(v) + f_X(v)f_Y(v) \end{aligned}$$

Problem 5.9.5

the bivariate Gaussian PDF as

$$f_{X,Y}(x,y) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2} \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-(y-\tilde{\mu}_Y(x))^2/2\tilde{\sigma}_Y^2}$$

where

$$\tilde{\mu}_Y(x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) \qquad \tilde{\sigma}_Y = \sigma_Y\sqrt{1-\rho^2}$$

However, the definitions of $\tilde{\mu}_Y(x)$ and $\tilde{\sigma}_Y$ are not particularly important for this exercise. When we integrate the joint PDF over all x and y, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy &= \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-(y-\tilde{\mu}_Y(x))^2/2\tilde{\sigma}_Y^2}\,dy}_{1}\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}\,dx \end{aligned}$$

The marked integral equals 1 because for each value of x, it is the integral of a Gaussian PDF of one variable over all possible values. In fact, it is the integral of the conditional PDF $f_{Y|X}(y|x)$ over all possible y. To complete the proof, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}\,dx = 1$$

since the remaining integral is the integral of the marginal Gaussian PDF $f_X(x)$ over all possible x.

Problem 5.10.4

Let A denote the event $X_n = \max(X_1, \dots, X_n)$. We can find $P[A]$ by conditioning on the value of X_n.

$$\begin{aligned}
P[A] &= P[X_1 \le X_n, X_2 \le X_n, \cdots, X_{n_1} \le X_n] \\
&= \int_{-\infty}^{\infty} P[X_1 < X_n, X_2 < X_n, \cdots, X_{n-1} < X_n | X_n = x] f_{X_n}(x)\, dx \\
&= \int_{-\infty}^{\infty} P[X_1 < x, X_2 < x, \cdots, X_{n-1} < x] f_X(x)\, dx
\end{aligned}$$

Since $X_1, \dots, X_{n-1}$ are iid,

$$\begin{aligned}
P[A] &= \int_{-\infty}^{\infty} P[X_1 \le x] P[X_2 \le x] \cdots P[X_{n-1} \le x] f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} [F_X(x)]^{n-1} f_X(x)\, dx \\
&= \frac{1}{n} [F_X(x)]^n \Big|_{-\infty}^{\infty} \\
&= \frac{1}{n}(1 - 0) \\
&= 1/n
\end{aligned}$$

Not surprisingly, since the X_i are identical, symmetry would suggest that X_n is as likely as any of the other X_i to be the largest. Hence $P[A] = 1/n$ should not be surprising.