Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,4.1.4 4.2.4 4.3.7 4.4.10 4.4.11 4.5.6 4.6.8 4.6.9 4.7.14 4.7.15 4.7.16 4.8.3 and 4.8.4

Problem 4.1.4

(a) By definition, $\lceil nx \rceil$ is the smallest integer that is greater than or equal to nx. This implies

$$nx \leq \lceil nx \rceil \leq nx + 1$$

(b) By part (a),

$$\frac{nx}{n} \leq \frac{\lceil nx \rceil}{n} \leq \frac{nx+1}{n}$$

That is,

$$x \leq \frac{\lceil nx \rceil}{n} \leq x + \frac{1}{n}$$

This implies

$$x \leq \lim_{n\to\infty} \frac{\lceil nx \rceil}{n} \leq \lim_{n\to\infty} x + \frac{1}{n} = x$$

Problem 4.2.4

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3)$$

For the PDF to be non-negative for $x \in [0,1]$, we must have $ax + 2 - 2a/3 \geq 0$ for all $x \in [0,1]$. This requirement can be written as

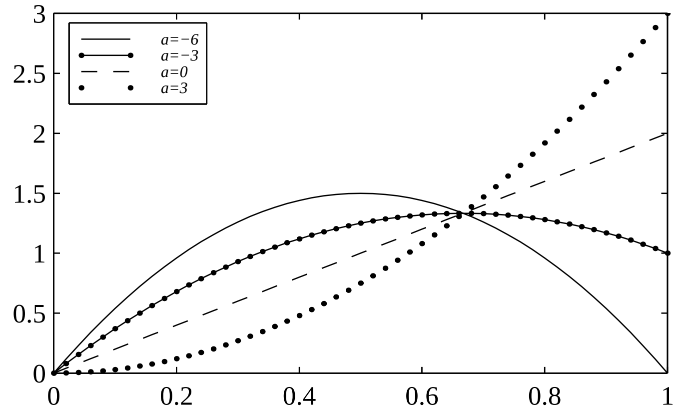
$$a(2/3 - x) \leq 2 \qquad (0 \leq x \leq 1)$$

For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \leq x < 2/3$ and $2/3 < x \leq 1$ separately because of the sign change of the inequality.

When $0 \le x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \le 2$ or $a \le 3$. When $2/3 < x \le 1$, we can write the constraint as $a(x - 2/3) \ge -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \ge -2$ or $a \ge -6$. Thus our a complete expression for our requirements are

$$-6 \le a \le 3 \qquad b = 2 - 2a/3$$

As we see in the following plot, the shape of the PDF $f_X(x)$ varies greatly with the value of a.



Problem 4.3.7

find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases}$$

(a) The expected value of U is

$$\begin{aligned} E[U] \int_{-\infty}^{\infty} u f_U(u)\, du &= \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \\ &= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} \\ &= -1 + 3 = 2 \end{aligned}$$

(b) The second moment of U is

$$\begin{aligned} E\left[U^2\right] \int_{-\infty}^{\infty} u^2 f_U(u)\, du &= \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \\ &= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} \\ &= 49/3 \end{aligned}$$

The variance of U is $\text{Var}[U] = E\left[U^2\right] - (E[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\,du = \int e^{(\ln 2)u}\,du = \frac{1}{\ln 2}e^{(\ln 2)u} = \frac{2^u}{\ln 2}$$

The expected value of 2^U is then

$$\begin{aligned} E\left[2^U\right] = \int_{-\infty}^{\infty} 2^u f_U(u)\,du &= \int_{-5}^{-3} \frac{2^u}{8}\,du + \int_{3}^{5} \frac{3 \cdot 2^u}{8}\,du \\ &= \left.\frac{2^u}{8\ln 2}\right|_{-5}^{-3} + \left.\frac{3\cdot 2^u}{8\ln 2}\right|_{3}^{5} \\ &= \frac{2307}{256\ln 2} = 13.001 \end{aligned}$$

Problem 4.4.10

For $n = 1$, we have the fact $E[X] = 1/\lambda$ that is given in the problem statement. Now we assume that $E\left[X^{n-1}\right] = (n-1)!/\lambda^{n-1}$. To complete the proof, we show that this implies that $E[X^n] = n!/\lambda^n$. Specifically, we write

$$E[X^n] = \int_0 x^n \lambda e^{-\lambda x}\,dx$$

Now we use the integration by parts formula $\int u\,dv = uv - \int v\,du$ with $u = x^n$ and $dv = \lambda e^{-\lambda x}\,dx$. This implies $du = nx^{n-1}\,dx$ and $v = -e^{-\lambda x}$ so that

$$\begin{aligned} E[X^n] &= \left. -x^n e^{-\lambda x}\right|_0^{\infty} + \int_0^{\infty} nx^{n-1}e^{-\lambda x}\,dx \\ &= 0 + \frac{n}{\lambda}\int_0^{\infty} x^{n-1}\lambda e^{-\lambda x}\,dx \\ &= \frac{n}{\lambda}E\left[X^{n-1}\right] \end{aligned}$$

By our induction hyothesis, $E\left[X^{n-1}\right] = (n-1)!/\lambda^{n-1}$ which implies

$$E[X^n] = n!/\lambda^n$$

Problem 4.4.11

(a) Since $f_X(x) \geq 0$ and $x \geq r$ over the entire integral, we can write

$$\int_r^{\infty} x f_X(x)\,dx \geq \int_r^{\infty} r f_X(x)\,dx = rP[X > r]$$

(b) We can write the expected value of X in the form

$$E[X] = \int_0^r x f_X(x)\,dx + \int_r^{\infty} x f_X(x)\,dx$$

Hence,

$$rP[X > r] \leq \int_r^\infty x f_X(x)\, dx = E[X] - \int_0^r x f_X(x)\, dx$$

Allowing r to approach infinity yields

$$\lim_{r\to\infty} rP[X > r] \leq E[X] - \lim_{r\to\infty} \int_0^r x f_X(x)\, dx = E[X] - E[X] = 0$$

Since $rP[X > r] \geq 0$ for all $r \geq 0$, we must have $\lim_{r\to\infty} rP[X > r] = 0$.

(c) We can use the integration by parts formula $\int u\, dv = uv - \int v\, du$ by defining $u = 1 - F_X(x)$ and $dv = dx$. This yields

$$\int_0^\infty [1 - F_X(x)]\, dx = x[1 - F_X(x)]|_0^\infty + \int_0^\infty x f_X(x)\, dx$$

By applying part (a), we now observe that

$$x[1 - F_X(x)]|_0^\infty = \lim_{r\to\infty} r[1 - F_X(r)] - 0 = \lim_{r\to\infty} rP[X > r]$$

By part (b), $\lim_{r\to\infty} rP[X > r] = 0$ and this implies $x[1 - F_X(x)]|_0^\infty = 0$. Thus,

$$\int_0^\infty [1 - F_X(x)]\, dx = \int_0^\infty x f_X(x)\, dx = E[X]$$

Problem 4.5.6

We are given that there are 100,000,000 men in the United States and 23,000 of them are at least 7 feet tall, and the heights of U.S men are independent Gaussian random variables with mean $5'10''$.

(a) Let H denote the height in inches of a U.S male. To find σ_X, we look at the fact that the probability that $P[H \geq 84]$ is the number of men who are at least 7 feet tall divided by the total number of men (the frequency interpretation of probability). Since we measure H in inches, we have

$$P[H \geq 84] = \frac{23,000}{100,000,000} = \Phi\left(\frac{70 - 84}{\sigma_X}\right) = 0.00023$$

Since $\Phi(-x) = 1 - \Phi(x) = Q(x)$,

$$Q(14/\sigma_X) = 2.3 \cdot 10^{-4}$$

From Table 4.2, this implies $14/\sigma_X = 3.5$ or $\sigma_X = 4$.

(b) The probability that a randomly chosen man is at least 8 feet tall is

$$P[H \geq 96] = Q\left(\frac{96 - 70}{4}\right) = Q(6.5)$$

Unfortunately, Table 4.2 doesn't include $Q(6.5)$, although it should be apparent that the probability is very small. In fact, $Q(6.5) = 4.0 \times 10^{-11}$.

(c) First we need to find the probability that a man is at least 7'6".

$$P[H \geq 90] = Q\left(\frac{90-70}{4}\right) = Q(5) \approx 3 \cdot 10^{-7} = \beta$$

Although Table 4.2 stops at $Q(4.99)$, if you're curious, the exact value is $Q(5) = 2.87 \cdot 10^{-7}$.

Now we can begin to find the probability that no man is at least 7'6". This can be modeled as 100,000,000 repetitions of a Bernoulli trial with parameter $1-\beta$. The probability that no man is at least 7'6" is

$$(1-\beta)^{100,000,000} = 9.4 \times 10^{-14}$$

(d) The expected value of N is just the number of trials multiplied by the probability that a man is at least 7'6".

$$E[N] \quad = \quad 100,000,000 \cdot \beta = 30$$

Problem 4.6.8

good, that is, no foul occurs. The CDF of D obeys

$$F_D(y) = P[D \leq y|G]P[G] + P[D \leq y|G^c]P[G^c]$$

Given the event G,

$$P[D \leq y|G] = P[X \leq y-60] = 1 - e^{-(y-60)/10} \quad (y \geq 60)$$

Of course, for $y < 60$, $P[D \leq y|G] = 0$. From the problem statement, if the throw is a foul, then $D = 0$. This implies

$$P[D \leq y|G^c] = u(y)$$

where $u(\cdot)$ denotes the unit step function. Since $P[G] = 0.7$, we can write

$$\begin{aligned} F_D(y) &= P[G]P[D \leq y|G] + P[G^c]P[D \leq y|G^c] \\ &= \begin{cases} 0.3u(y) & y < 60 \\ 0.3 + 0.7(1 - e^{-(y-60)/10}) & y \geq 60 \end{cases} \end{aligned}$$

Another way to write this CDF is

$$F_D(y) = 0.3u(y) + 0.7u(y-60)(1 - e^{-(y-60)/10})$$

However, when we take the derivative, either expression for the CDF will yield the PDF. However, taking the derivative of the first expression perhaps may be simpler:

$$f_D(y) = \begin{cases} 0.3\delta(y) & y < 60 \\ 0.07e^{-(y-60)/10} & y \geq 60 \end{cases}$$

Taking the derivative of the second expression for the CDF is a little tricky because of the product of the exponential and the step function. However, applying the usual rule for the differentation of a product does give the correct answer:

$$\begin{aligned} f_D(y) &= 0.3\delta(y) + 0.7\delta(y-60)(1 - e^{-(y-60)/10}) + 0.07u(y-60)e^{-(y-60)/10} \\ &= 0.3\delta(y) + 0.07u(y-60)e^{-(y-60)/10} \end{aligned}$$

The middle term $\delta(y-60)(1 - e^{-(y-60)/10})$ dropped out because at $y = 60$, $e^{-(y-60)/10} = 1$.

Problem 4.6.9

The professor is on time and lectures the full 80 minutes with probability 0.7. That is, $P[T = 80] = 0.7$. Likewise when the professor is more than 5 minutes late, the students leave and a 0 minute lecture is observed. Since he is late 30% of the time and given that he is late, his arrival is uniformly distributed between 0 and 10 minutes, the probability that there is no lecture is

$$P[T = 0] = (0.3)(0.5) = 0.15$$

The only other possible lecture durations are uniformly distributed between 75 and 80 minutes, because the students will not wait longer then 5 minutes, and that probability must add to a total of $1 - 0.7 - 0.15 = 0.15$. So the PDF of T can be written as

$$f_T(t) = \begin{cases} 0.15\delta(t) & t = 0 \\ 0.03 & 75 \leq 7 < 80 \\ 0.7\delta(t - 80) & t = 80 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4.7.14

We can prove the assertion by considering the cases where $a > 0$ and $a < 0$, respectively. For the case where $a > 0$ we have

$$F_Y(y) \quad = \quad P[Y \leq y] = P\left[X \leq \frac{y - b}{a}\right] = F_X\left(\frac{y - b}{a}\right)$$

Therefore by taking the derivative we find that

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y - b}{a}\right) \qquad a > 0$$

Similarly for the case when $a < 0$ we have

$$F_Y(y) = P[Y \leq y] = P\left[X \geq \frac{y - b}{a}\right] = 1 - F_X\left(\frac{y - b}{a}\right)$$

And by taking the derivative, we find that for negative a,

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y - b}{a}\right) \qquad a < 0$$

A valid expression for both positive and negative a is

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right)$$

Therefore the assertion is proved.

Problem 4.7.15

Understanding this claim may be harder than completing the proof. Since $0 \leq F(x) \leq 1$, we know that $0 \leq U \leq 1$. This implies $F_U(u) = 0$ for $u < 0$ and $F_U(u) = 1$ for $u \geq 1$. Moreover, since $F(x)$ is an increasing function, we can write for $0 \leq u \leq 1$,

$$F_U(u) = P[F(X) \leq u] = P\left[X \leq F^{-1}(u)\right] = F_X\left(F^{-1}(u)\right)$$

Since $F_X(x) = F(x)$, we have for $0 \leq u \leq 1$,

$$F_U(u) = F(F^{-1}(u)) = u$$

Hence the complete CDF of U is

$$F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \leq u < 1 \\ 1 & u \geq 1 \end{cases}$$

That is, U is a uniform $[0,1]$ random variable.

Problem 4.7.16

First, we must verify that $F^{-1}(u)$ is a nondecreasing function. To show this, suppose that for $u \geq u'$, $x = F^{-1}(u)$ and $x' = F^{-1}(u')$. In this case, $u = F(x)$ and $u' = F(x')$. Since $F(x)$ is nondecreasing, $F(x) \geq F(x')$ implies that $x \geq x'$. Hence, we can write

$$F_X(x) = P\left[F^{-1}(U) \leq x\right] = P[U \leq F(x)] = F(x)$$

Problem 4.8.3

W is

$$f_W(w) = \frac{1}{\sqrt{32\pi}} e^{-w^2/32}$$

(a) Since W has expected value $\mu = 0$, $f_W(w)$ is symmetric about $w = 0$. Hence $P[C] = P[W > 0] = 1/2$. From Definition 4.15, the conditional PDF of W given C is

$$f_{W|C}(w) = \begin{cases} f_W(w)/P[C] & w \in C \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2e^{-w^2/32}/\sqrt{32\pi} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) The conditional expected value of W given C is

$$E[W|C] = \int_{-\infty}^{\infty} w f_{W|C}(w)\, dw = \frac{2}{4\sqrt{2\pi}} \int_0^{\infty} we^{-w^2/32}\, dw$$

Making the substitution $v = w^2/32$, we obtain

$$E[W|C] = \frac{32}{\sqrt{32\pi}} \int_0^{\infty} e^{-v}\, dv = \frac{32}{\sqrt{32\pi}}$$

(c) The conditional second moment of W is

$$E\left[W^2|C\right] = \int_{-\infty}^{\infty} w^2 f_{W|C}(w)\, dw = 2\int_0^{\infty} w^2 f_W(w)\, dw$$

We observe that $w^2 f_W(w)$ is an even function. Hence

$$E\left[W^2|C\right] = 2\int_0^{\infty} w^2 f_W(w)\, dw = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = E\left[W^2\right] = \sigma^2 = 16$$

Lastly, the conditional variance of W given C is

$$\text{Var}[W|C] = E\left[W^2|C\right] - (E[W|C])^2 = 16 - 32/\pi = 5.81$$

Problem 4.8.4

(a) To find the conditional moments, we first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\, dt = -e^{-100t}\Big|_{0.02}^{\infty} = e^{-2}$$

From Definition 4.15, the conditional PDF of T is

$$f_{T|T>0.02}(t) = \begin{cases} \frac{f_T(t)}{P[T>0.02]} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 100e^{-100(t-0.02)} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases}$$

The conditional mean of T is

$$E[T|T > 0.02] = \int_{0.02}^{\infty} t(100)e^{-100(t-0.02)}\, dt$$

The substitution $\tau = t - 0.02$ yields

$$\begin{aligned} E[T|T > 0.02] &= \int_0^{\infty} (\tau + 0.02)(100)e^{-100\tau}\, d\tau \\ &= \int_0^{\infty} (\tau + 0.02) f_T(\tau)\, d\tau \\ &= E[T + 0.02] = 0.03 \end{aligned}$$

(b) The conditional second moment of T is

$$E\left[T^2|T > 0.02\right] = \int_{0.02}^{\infty} t^2(100)e^{-100(t-0.02)}\, dt$$

The substitution $\tau = t - 0.02$ yields

$$\begin{aligned} E\left[T^2 | T > 0.02\right] &= \int_0^\infty (\tau + 0.02)^2 (100) e^{-100\tau} \, d\tau \\ &= \int_0^\infty (\tau + 0.02)^2 f_T(\tau) \, d\tau \\ &= E\left[(T + 0.02)^2\right] \end{aligned}$$

Now we can calculate the conditional variance.

$$\begin{aligned} \mathrm{Var}[T | T > 0.02] &= E\left[T^2 | T > 0.02\right] - (E[T | T > 0.02])^2 \\ &= E\left[(T + 0.02)^2\right] - (E[T + 0.02])^2 \\ &= \mathrm{Var}[T + 0.02] \\ &= \mathrm{Var}[T] = 0.01 \end{aligned}$$