Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,3.1.6 3.1.7 3.2.5 3.3.5 3.4.8 3.5.1 3.6.7 3.6.8 3.6.9 3.7.8 3.8.5 and 3.8.6

Problem 3.1.6

The joint PMF of X and N is $P_{N,X}(n,x) = P[N = n, X = x]$, which is the probability that $N = n$ and $X = x$. This means that both events must be satisfied. The approach we use is similar to that used in finding the Pascal PMF in Example 2.15. Since X can take on only the two values 0 and 1, let's consider each in turn. When $X = 0$ that means that a rejection occurred on the last test and that the other $n-1$ rejections must have occurred in the previous $r-1$ tests. Thus,

$$P_{N,X}(n,0) = \binom{r-1}{n-1}(1-p)^{n-1}p^{r-1-(n-1)}(1-p) \qquad n = 1,\ldots,r$$

When $X = 1$ the last test was acceptable and therefore we know that the $N = n \le r-1$ tails must have occurred in the previous $r-1$ tests. In this case,

$$P_{N,X}(n,1) = \binom{r-1}{n}(1-p)^{n}p^{r-1-n}p \qquad n = 0,\ldots,r-1$$

We can combine these cases into a single complete expression for the joint PMF.

$$P_{X,N}(x,n) = \begin{cases} \binom{r-1}{n-1}(1-p)^n p^{r-n} & x = 0, n = 1,2,\ldots,r \\ \binom{r-1}{n}(1-p)^n p^{r-n} & x = 1, n = 0,1,\ldots,r-1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.1.7

Each circuit test produces an acceptable circuit with probability p. Let N denote the number of rejected circuits that occur in r tests and X is the number of acceptable circuits before the first reject. The joint PMF, $P_{N,X}(n,x) = P[N = n, X = x]$ can be found by realizing that $\{N = n, X = x\}$ occurs if and only if the following events occur:

A The first x tests must be acceptable.

B Test $x+1$ must be a rejection since otherwise we would have $x+1$ acceptable at the beginnning.

C The remaining $r-x-1$ tests must contain $n-1$ rejections.

Since the events A, B and C are independent, the joint PMF for $x+n \le r$, $x \ge 0$ and $n \ge 0$ is

$$P_{N,X}(n,x) = \underbrace{p^x}_{P[A]}\underbrace{(1-p)}_{P[B]}\underbrace{\binom{r-x-1}{n-1}(1-p)^{n-1}p^{r-x-1-(n-1)}}_{P[C]}$$

After simplifying, a complete expression for the joint PMF is

$$P_{N,X}(n,x) = \begin{cases} \binom{r-x-1}{n-1}p^{r-n}(1-p)^n & x+n \le r, x \ge 0, n \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.2.5

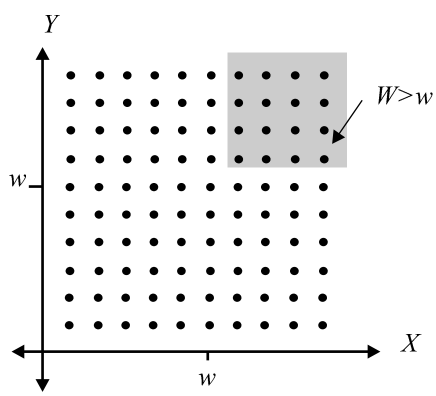
N is

$$P_N(n) = \sum_k P_{N,K}(n,k) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}$$

For $k = 0, 1, \ldots$, the marginal PMF of K is

$$\begin{aligned} P_K(k) &= \sum_{n=k}^{\infty} \frac{100^n e^{-100}}{(n+1)!} \\ &= \frac{1}{100} \sum_{n=k}^{\infty} \frac{100^{n+1} e^{-100}}{(n+1)!} \\ &= \frac{1}{100} \sum_{n=k}^{\infty} P_N(n+1) \\ &= P[N > k]/100 \end{aligned}$$

Problem 3.3.5



The x, y pairs with nonzero probability are shown in the figure at right. From the figure, we observe that for $w = 0, 1, \ldots, 10$,

$$\begin{aligned} P[W > w] &= P[\min(X,Y) > w] \\ &= P[X > w, Y > w] \\ &= 0.01(10 - w)^2 \end{aligned}$$

To find the PMF of W, we observe that for $w = 1, \ldots, 10$,

$$\begin{aligned} P_W(w) &= P[W > w - 1] - P[W > w] \\ &= 0.01[(10 - w - 1)^2 - (10 - w)^2] \\ &= 0.01(21 - 2w) \end{aligned}$$

The complete expression for the PMF of W is

$$P_W(w) = \begin{cases} 0.01(21 - 2w) & w = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.4.8

For this problem, calculating the marginal PMF of K is not easy. However, the marginal PMF of N is easy to find. For $n = 1, 2, \ldots$,

$$P_N(n) = \sum_{k=1}^{n} \frac{(1-p)^{n-1} p}{n} = (1-p)^{n-1} p$$

That is, N has a geometric PMF. From Appendix A, we note that

$$E[N] = \frac{1}{p} \qquad \mathrm{Var}[N] = \frac{1-p}{p^2}$$

We can use these facts to find the second moment of N.

$$E\left[N^2\right] = \mathrm{Var}[N] + (E[N])^2 = \frac{2-p}{p^2}$$

Now we can calculate the moments of K.

$$E[K] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} k \frac{(1-p)^{n-1} p}{n} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1} p}{n} \sum_{k=1}^{n} k$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[K] = \sum_{n=1}^{\infty} \frac{n+1}{2} (1-p)^{n-1} p = E\left[\frac{N+1}{2}\right] = \frac{1}{2p} + \frac{1}{2}$$

We now can calculate the sum of the moments.

$$E[N+K] = E[N] + E[K] = \frac{3}{2p} + \frac{1}{2}$$

The second moment of K is

$$E\left[K^2\right] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} k^2 \frac{(1-p)^{n-1} p}{n} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1} p}{n} \sum_{k=1}^{n} k^2$$

Using the identity $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, we obtain

$$E\left[K^2\right] = \sum_{n=1}^{\infty} \frac{(n+1)(2n+1)}{6} (1-p)^{n-1} p = E\left[\frac{(N+1)(2N+1)}{6}\right]$$

Applying the values of $E[N]$ and $E\left[N^2\right]$ found above, we find that

$$E\left[K^2\right] = \frac{E\left[N^2\right]}{3} + \frac{E[N]}{2} + \frac{1}{6} = \frac{2}{3p^2} + \frac{1}{6p} + \frac{1}{6}$$

Thus, we can calculate the variance of K.

$$\mathrm{Var}[K] = E\left[K^2\right] - (E[K])^2 = \frac{5}{12p^2} - \frac{1}{3p} + \frac{5}{12}$$

To find the correlation of N and K,

$$E[NK] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} nk \frac{(1-p)^{n-1} p}{n} = \sum_{n=1}^{\infty} (1-p)^{n-1} p \sum_{k=1}^{n} k$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[NK] = \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (1-p)^{n-1} p = E\left[\frac{N(N+1)}{2}\right] = \frac{1}{p^2}$$

Finally, the covariance is

$$\mathrm{Cov}[N,K] = E[NK] - E[N]E[K] = \frac{1}{2p^2} - \frac{1}{2p}$$

Problem 3.5.1

The event A occurs iff $X > 5$ and $Y > 5$ and has probability

$$P[A] = P[X > 5, Y > 5] = \sum_{x=6}^{10} \sum_{y=6}^{10} 0.01 = 0.25$$

From Theorem 3.11,

$$P_{X,Y|A}(x,y) = \begin{cases} \frac{P_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.04 & x = 6,\ldots,10; y = 6,\ldots,20 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.6.7

The key to solving this problem is to find the joint PMF of M and N. Note that $N \geq M$. For $n > m$, the joint event $\{M = m, N = n\}$ has probability

$$\begin{aligned} P[M = m, N = n] &= P[\overbrace{dd\cdots d}^{\substack{m-1 \\ \text{calls}}} v \overbrace{dd\cdots d}^{\substack{n-m-1 \\ \text{calls}}} v] \\ &= (1-p)^{m-1} p (1-p)^{n-m-1} p \\ &= (1-p)^{n-2} p^2 \end{aligned}$$

A complete expression for the joint PMF of M and N is

$$P_{M,N}(m,n) = \begin{cases} (1-p)^{n-2} p^2 & m = 1,2,\ldots,n-1;\ n = m+1, m+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

For $n = 2, 3, \ldots$, the marginal PMF of N satisfies

$$P_N(n) = \sum_{m=1}^{n-1} (1-p)^{n-2} p^2 = (n-1)(1-p)^{n-2} p^2$$

Similarly, for $m = 1, 2, \ldots$, the marginal PMF of M satisfies

$$\begin{aligned} P_M(m) &= \sum_{n=m+1}^{\infty} (1-p)^{n-2} p^2 \\ &= p^2[(1-p)^{m-1} + (1-p)^m + \cdots] \\ &= (1-p)^{m-1} p \end{aligned}$$

The complete expressions for the marginal PMF's are

$$P_M(m) = \begin{cases} (1-p)^{m-1} p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_N(n) = \begin{cases} (n-1)(1-p)^{n-2} p^2 & n = 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also, N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find.

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{n-m-1}p & n = m+1, m+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The interpretation of the conditional PMF of N given M is that given $M = m$, $N = m + N'$ where N' has a geometric PMF with mean $1/p$. The conditional PMF of M given N is

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} 1/(n-1) & m = 1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Given that call $N = n$ was the second voice call, the first voice call is equally likely to occur in any of the previous $n-1$ calls.

Problem 3.6.8

(a) The number of buses, N, must be greater than zero. Also, the number of minutes that pass cannot be less than the number of buses. Thus, $P[N = n, T = t] > 0$ for integers n, t satisfying $1 \le n \le t$.

(b) First, we find the joint PMF of N and T by carefully considering the possible sample paths. In particular, $P_{N,T}(n,t) = P[ABC] = P[A]P[B]P[C]$ where the events A, B and C are

$$\begin{aligned} A &= \{n-1 \text{ buses arrive in the first } t-1 \text{ minutes}\} \\ B &= \{\text{none of the first } n-1 \text{ buses are boarded}\} \\ C &= \{\text{at time } t \text{ a bus arrives and is boarded}\} \end{aligned}$$

These events are independent since each trial to board a bus is independent of when the buses arrive. These events have probabilities

$$\begin{aligned} P[A] &= \binom{t-1}{n-1} p^{n-1}(1-p)^{t-1-(n-1)} \\ P[B] &= (1-q)^{n-1} \\ P[C] &= pq \end{aligned}$$

Consequently, the joint PMF of N and T is

$$P_{N,T}(n,t) = \begin{cases} \binom{t-1}{n-1} p^{n-1}(1-p)^{t-n}(1-q)^{n-1}pq & n \ge 1, t \ge n \\ 0 & \text{otherwise} \end{cases}$$

(c) It is possible to find the marginal PMF's by summing the joint PMF. However, it is much easier to obtain the marginal PMFs by consideration of the experiment. Specifically, when a bus arrives, it is boarded with probability q. Moreover, the experiment ends when a bus is boarded. By viewing whether each arriving bus is boarded as an independent trial, N is the number of trials until the first success. Thus, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-q)^{n-1}q & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

To find the PMF of T, suppose we regard each minute as an independent trial in which a success occurs if a bus arrives and that bus is boarded. In this case, the success probability is pq and T is the number of minutes up to and including the first success. The PMF of T is also geometric.

$$P_T(t) = \begin{cases} (1-pq)^{t-1}pq & t = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

(d) Once we have the marginal PMFs, the conditional PMFs are easy to find.

$$P_{N|T}(n|t) = \frac{P_{N,T}(n,t)}{P_T(t)} = \begin{cases} \binom{t-1}{n-1}\left(\frac{p(1-q)}{1-pq}\right)^{n-1}\left(\frac{1-p}{1-pq}\right)^{t-1-(n-1)} & n = 1,2,\ldots,t \\ 0 & \text{otherwise} \end{cases}$$

That is, given you depart at time $T = t$, the number of buses that arrive during minutes $1,\ldots,t-1$ has a binomial PMF since in each minute a bus arrives with probability p. Similarly, the conditional PMF of T given N is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \binom{t-1}{n-1}p^n(1-p)^{t-n} & t = n,n+1,\ldots \\ 0 & \text{otherwise} \end{cases}$$

This result can be explained. Given that you board bus $N = n$, the time T when you leave is the time for n buses to arrive. If we view each bus arrival as a success of an independent trial, the time for n buses to arrive has the above Pascal PMF.

Problem 3.6.9

what type of call (if any) that arrived in any 1 millisecond period, it will be apparent that a fax call arrives with probability $\alpha = pqr$ or no fax arrives with probability $1-\alpha$. That is, whether a fax message arrives each millisecond is a Bernoulli trial with success probability α. Thus, the time required for the first success has the geometric PMF

$$P_T(t) = \begin{cases} (1-\alpha)^{t-1}\alpha & t = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Note that N is the number of trials required to observe 100 successes. Moreover, the number of trials needed to observe 100 successes is $N = T + N'$ where N' is the number of trials needed to observe successes 2 through 100. Since N' is just the number of trials needed to observe 99 successes, it has the Pascal PMF

$$P_{N'}(n) = \begin{cases} \binom{n-1}{98}\alpha^{98}(1-\alpha)^{n-98} & n = 99,100,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Since the trials needed to generate successes 2 though 100 are independent of the trials that yield the first success, N' and T are independent. Hence

$$P_{N|T}(n|t) = P_{N'|T}(n-t|t) = P_{N'}(n-t)$$

Applying the PMF of N' found above, we have

$$P_{N|T}(n|t) = \begin{cases} \binom{n-1}{98}\alpha^{98}(1-\alpha)^{n-t-98} & n = 99+t,100+t,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Finally the joint PMF of N and T is

$$P_{N,T}(n,t) = P_{N|T}(n|t)\,P_T(t)$$
$$= \begin{cases} \binom{n-t-1}{98}\alpha^{99}(1-\alpha)^{n-99}\alpha & t = 1,2,\ldots; n = 99+t, 100+t,\ldots \\ 0 & \text{otherwise} \end{cases}$$

This solution can also be found a consideration of the sample sequence of Bernoulli trials in which we either observe or do not observe a fax message. To find the conditional PMF $P_{T|N}(t|n)$, we first must recognize that N is simply the number of trials needed to observe 100 successes and thus has the Pascal PMF

$$P_N(n) = \begin{cases} \binom{n-1}{99}\alpha^{100}(1-\alpha)^{n-100} & n = 100, 101, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Hence the conditional PMF is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \frac{\binom{n-t-1}{98}}{\binom{n-1}{99}}\,\frac{1-\alpha}{\alpha}$$

Problem 3.7.8

The key to this problem is understanding that "Factory Q" and "Factory R" are synonyms for $M = 60$ and $M = 180$. Similarly, "small", "medium", and "large" orders correspond to the events $B = 1$, $B = 2$ and $B = 3$.

(a) The following table given in the problem statement

	Factory Q	Factory R
small order	0.3	0.2
medium order	0.1	0.2
large order	0.1	0.1

can be translated into the following joint PMF for B and M.

$P_{B,M}(b,m)$	$m = 60$	$m = 180$
$b = 1$	0.3	0.2
$b = 2$	0.1	0.2
$b = 3$	0.1	0.1

(b) Before we find $E[B]$, it will prove helpful for the remainder of the problem to find thhe marginal PMFs $P_B(b)$ and $P_M(m)$. These can be found from the row and column sums of the table of the joint PMF

$P_{B,M}(b,m)$	$m = 60$	$m = 180$	$P_B(b)$
$b = 1$	0.3	0.2	0.5
$b = 2$	0.1	0.2	0.3
$b = 3$	0.1	0.1	0.2
$P_M(m)$	0.5	0.5	

The expected number of boxes is

$$E[B] = \sum_b b P_B(b) = 1(0.5) + 2(0.3) + 3(0.2) = 1.7$$

(c) From the marginal PMF of B, we know that $P_B(2) = 0.3$. The conditional PMF of M given $B = 2$ is

$$P_{M|B}(m|2) = \frac{P_{B,M}(2,m)}{P_B(2)} = \begin{cases} 1/3 & m = 60 \\ 2/3 & m = 180 \\ 0 & \text{otherwise} \end{cases}$$

(d) The conditional expectation of M given $B = 2$ is

$$E[M|B=2] = \sum_m m P_{M|B}(m|2) = 60(1/3) + 180(2/3) = 140$$

(e) From the marginal PMFs we calculated in the table of part (b), we can conclude that B and M are not independent. since $P_{B,M}(1,60) \neq P_B(1) P_M(m) 60$.

(f) In terms of M and B, the cost (in cents) of sending a shipment is $C = BM$. The expected value of C is

$$\begin{aligned} E[C] &= \sum_{b,m} bm P_{B,M}(b,m) \\ &= 1(60)(0.3) + 2(60)(0.1) + 3(60)(0.1) \\ &\quad + 1(180)(0.2) + 2(180)(0.2) + 3(180)(0.1) = 210 \end{aligned}$$

Problem 3.8.5

is added to the jackpot,

$$J_{i-1} = J_i + \frac{N_i}{2}$$

Given $J_i = j$, N_i has a Poisson distribution with mean j. so that $E[N_i|J_i = j] = j$ and that $\mathrm{Var}[N_i|J_i = j] = j$. This implies

$$E\left[N_i^2|J_i = j\right] = \mathrm{Var}[N_i|J_i = j] + (E[N_i|J_i = j])^2 = j + j^2$$

In terms of the conditional expectations given J_i, these facts can be written as

$$E[N_i|J_i] = J_i \qquad E\left[N_i^2|J_i\right] = J_i + J_i^2$$

This permits us to evaluate the moments of J_{i-1} in terms of the moments of J_i. Specifically,

$$E[J_{i-1}|J_i] = E[J_i|J_i] + \frac{1}{2}E[N_i|J_i] = J_i + \frac{J_i}{2} = \frac{3J_i}{2}$$

This implies

$$E[J_{i-1}] = \frac{3}{2}E[J_i]$$

We can use this the calculate $E[J_i]$ for all i. Since the jackpot starts at 1 million dollars, $J_6 = 10^6$ and $E[J_6] = 10^6$. This implies

$$E[J_i] = (3/2)^{6-i} 10^6$$

Now we will find the second moment $E\left[J_i^2\right]$. Since $J_{i-1}^2 = J_i^2 + N_iJ_i + N_i^2/4$, we have

$$\begin{aligned}
E\left[J_{i-1}^2|J_i\right] &= E\left[J_i^2|J_i\right] + E[N_iJ_i|J_i] + E\left[N_i^2|J_i\right]/4 \\
&= J_i^2 + J_iE[N_i|J_i] + (J_i + J_i^2)/4 \\
&= (3/2)^2 J_i^2 + J_i/4
\end{aligned}$$

By taking the expectation over J_i we have

$$E\left[J_{i-1}^2\right] = (3/2)^2 E\left[J_i^2\right] + E[J_i]/4$$

This recursion allows us to calculate $E\left[J_i^2\right]$ for $i = 6, 5, \ldots, 0$. Since $J_6 = 10^6$, $E\left[J_6^2\right] = 10^{12}$. From the recursion, we obtain

$$\begin{aligned}
E\left[J_5^2\right] &= (3/2)^2 E\left[J_6^2\right] + E[J_6]/4 = (3/2)^2 10^{12} + \frac{1}{4}10^6 \\
E\left[J_4^2\right] &= (3/2)^2 E\left[J_5^2\right] + E[J_5]/4 = (3/2)^4 10^{12} + \frac{1}{4}\left[(3/2)^2 + (3/2)\right]10^6 \\
E\left[J_3^2\right] &= (3/2)^2 E\left[J_4^2\right] + E[J_4]/4 = (3/2)^6 10^{12} + \frac{1}{4}\left[(3/2)^4 + (3/2)^3 + (3/2)^2\right]10^6
\end{aligned}$$

The same recursion will also allow us to show that

$$\begin{aligned}
E\left[J_2^2\right] &= (3/2)^8 10^{12} + \frac{1}{4}\left[(3/2)^6 + (3/2)^5 + (3/2)^4 + (3/2)^3\right]10^6 \\
E\left[J_1^2\right] &= (3/2)^{10} 10^{12} + \frac{1}{4}\left[(3/2)^8 + (3/2)^7 + (3/2)^6 + (3/2)^5 + (3/2)^4\right]10^6 \\
E\left[J_0^2\right] &= (3/2)^{12} 10^{12} + \frac{1}{4}\left[(3/2)^{10} + (3/2)^9 + \cdots + (3/2)^5\right]10^6
\end{aligned}$$

Finally, day 0 is the same as any other day in that $J = J_0 + N_0/2$ where N_0 is a Poisson random variable with mean J_0. By the same argument that we used to develop recursions for $E[J_i]$ and $E\left[J_i^2\right]$, we can show

$$E[J] = (3/2)E[J_0] = (3/2)^7 10^6 \approx 17 \times 10^6$$

and

$$\begin{aligned}
E\left[J^2\right] &= (3/2)^2 E\left[J_0^2\right] + E[J_0]/4 \\
&= (3/2)^{14} 10^{12} + \frac{1}{4}\left[(3/2)^{12} + (3/2)^{11} + \cdots + (3/2)^6\right]10^6 \\
&= (3/2)^{14} 10^{12} + \frac{10^6}{2}(3/2)^6[(3/2)^7 - 1]
\end{aligned}$$

Finally, the variance of J is

$$\mathrm{Var}[J] = E\left[J^2\right] - (E[J])^2 = \frac{10^6}{2}(3/2)^6[(3/2)^7 - 1]$$

Since the variance is hard to interpret, we note that the standard deviation of J is $\sigma_J \approx 9572$. Although the expected jackpot grows rapidly, the standard deviation of the jackpot is fairly small.

Problem 3.8.6

(a) The sample space is

$$S_{X,Y,Z} = \{(x,y,z)|x+y+z=5, x\geq 0, y\geq 0, z\geq 0, x, y, z \text{ integer}\}$$
$$= \left\{\begin{array}{llllll} (0,0,5), & & & & & \\ (0,1,4), & (1,0,4), & & & & \\ (0,2,3), & (1,1,3), & (2,0,3), & & & \\ (0,3,2), & (1,2,2), & (2,1,2), & (3,0,2), & & \\ (0,4,1), & (1,3,1), & (2,2,1), & (3,1,1), & (4,0,1), & \\ (0,5,0), & (1,4,0), & (2,3,0), & (3,2,0), & (4,1,0), & (5,0,0) \end{array}\right\}$$

(b) As we see in the above list of elements of $S_{X,Y,Z}$, just writing down all the elements is not so easy. Similarly, representing the joint PMF is usually not very straightforward. Here are the probabilities in a list.

(x,y,z)	$P_{X,Y,Z}(x,y,z)$	$P_{X,Y,Z}(x,y,z)$ (decimal)
$(0,0,5)$	$(1/6)^5$	1.29×10^{-4}
$(0,1,4)$	$5(1/2)(1/6)^4$	1.93×10^{-3}
$(1,0,4)$	$5(1/3)(1/6)^4$	1.29×10^{-3}
$(0,2,3)$	$10(1/2)^2(1/6)^3$	1.16×10^{-2}
$(1,1,3)$	$20(1/3)(1/2)(1/6)^3$	1.54×10^{-2}
$(2,0,3)$	$10(1/3)^2(1/6)^3$	5.14×10^{-3}
$(0,3,2)$	$10(1/2)^3(1/6)^2$	3.47×10^{-2}
$(1,2,2)$	$30(1/3)(1/2)^2(1/6)^2$	6.94×10^{-2}
$(2,1,2)$	$30(1/3)^2(1/2)(1/6)^2$	4.63×10^{-2}
$(3,0,2)$	$10(1/2)^3(1/6)^2$	1.03×10^{-2}
$(0,4,1)$	$5(1/2)^4(1/6)$	5.21×10^{-2}
$(1,3,1)$	$20(1/3)(1/2)^3(1/6)$	1.39×10^{-1}
$(2,2,1)$	$30(1/3)^2(1/2)^2(1/6)$	1.39×10^{-1}
$(3,1,1)$	$20(1/3)^3(1/2)(1/6)$	6.17×10^{-2}
$(4,0,1)$	$5(1/3)^4(1/6)$	1.03×10^{-2}
$(0,5,0)$	$(1/2)^5$	3.13×10^{-2}
$(1,4,0)$	$5(1/3)(1/2)^4$	1.04×10^{-1}
$(2,3,0)$	$10(1/3)^2(1/2)^3$	1.39×10^{-1}
$(3,2,0)$	$10(1/3)^3(1/2)^2$	9.26×10^{-2}
$(4,1,0)$	$5(1/3)^4(1/2)$	3.09×10^{-2}
$(5,0,0)$	$(1/3)^5$	4.12×10^{-3}

(c) Note that Z is the number of three page faxes. In principle, we can sum the joint PMF $P_{X,Y,Z}(x,y,z)$ over all x,y to find $P_Z(z)$. However, it is better to realize that each fax has 3 pages with probability $1/6$, independent of any other fax. Thus, Z has the binomial PMF

$$P_Z(z) = \begin{cases} \binom{5}{z}(1/6)^z(5/6)^{5-z} & z=0,1,\ldots,5 \\ 0 & \text{otherwise} \end{cases}$$

(d) From the properties of the binomial distribution given in Appendix A, we know that $E[Z] = 5(1/6)$.

(e) We want to find the conditional PMF of the number X of 1-page faxes and number Y of 2-page faxes given $Z = 2$ 3-page faxes. Note that given $Z = 2$, $X + Y = 3$. Hence for non-negative integers x, y satisfying $x + y = 3$,

$$P_{X,Y|Z}(x,y|2) = \frac{P_{X,Y,Z}(x,y,2)}{P_Z(2)} = \frac{\frac{5!}{x!y!2!}(1/3)^x(1/2)^y(1/6)^2}{\binom{5}{2}(1/6)^2(5/6)^3}$$

With some algebra, the complete expression of the conditional PMF is

$$P_{X,Y|Z}(x,y|2) = \begin{cases} \frac{3!}{x!y!}(2/5)^x(3/5)^y & x+y=3, x\geq 0, y\geq 0; x,y \text{ integer} \\ 0 & \text{otherwise} \end{cases}$$

To interpret the above expression, we observe that if $Z = 2$, then $Y = 3 - X$ and

$$P_{X|Z}(x|2) = P_{X,Y|Z}(x,3-x|2) = \begin{cases} \binom{3}{x}(2/5)^x(3/5)^{3-x} & x = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

That is, given $Z = 2$, there are 3 faxes left, each of which independently could be a 1-page fax. The conditonal PMF of the number of 1-page faxes is binomial where $2/5$ is the conditional probability that a fax has 1 page given that it either has 1 page or 2 pages. Moreover given $X = x$ and $Z = 2$ we must have $Y = 3 - x$.

(f) Given $Z = 2$, the conditional PMF of X is binomial for 3 trials and success probability $2/5$. The conditional expectation of X givn $Z = 2$ is $E[X|Z = 2] = 3(2/5) = 6/5$.

(g) There are several ways to solve this problem. The most straightforward approach is to realize that for integers $0 \leq x \leq 5$ and $0 \leq y \leq 5$, the event $\{X = x, Y = y\}$ occurs iff $\{X = x, Y = y, Z = 5 - (x + y)\}$. For the rest of this problem, we assume x and y are non-negative integers so that

$$\begin{aligned} P_{X,Y}(x,y) &= P_{X,Y,Z}(x,y,5-(x+y)) \\ &= \begin{cases} \frac{5!}{x!y!(5-x-y)!}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{5-x-y} & 0 \leq x+y \leq 5, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Tha above expression may seem unwieldy and it isn't even clear that it will sum to 1. To simplify the expression, we observe that

$$P_{X,Y}(x,y) = P_{X,Y,Z}(x,y,5-x-y) = P_{X,Y|Z}(x,y|5-x+y)\,P_Z(5-x-y)$$

Using $P_Z(z)$ found in part (c), we can calculate $P_{X,Y|Z}(x,y|5-x-y)$ for $0 \leq x+y \leq 5$. integer valued.

$$\begin{aligned} P_{X,Y|Z}(x,y|5-x+y) &= \frac{P_{X,Y,Z}(x,y,5-x-y)}{P_Z(5-x-y)} \\ &= \binom{x+y}{x}\left(\frac{1/3}{1/2+1/3}\right)^x\left(\frac{1/2}{1/2+1/3}\right)^y \\ &= \binom{x+y}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{(x+y)-x} \end{aligned}$$

In the above expression, it is wise to think of $x+y$ as some fixed value. In that case, we see that given $x+y$ is a fixed value, X and Y have a joint PMF given by a binomial distribution in x. This should not be surprising since it is just a generalization of the case when $Z=2$. That is, given that there were a fixed number of faxes that had either one or two pages, each of those faxes is a one page fax with probability $(1/3)/(1/2+1/3)$ and so the number of one page faxes should have a binomial distribution, Moreover, given the number X of one page faxes, the number Y of two page faxes is completely specified. Finally, by rewriting $P_{X,Y}(x,y)$ given above, the complete expression for the joint PMF of X and Y is

$$P_{X,Y}(x,y) = \begin{cases} \binom{5}{5-x-y}\left(\frac{1}{6}\right)^{5-x-y}\left(\frac{5}{6}\right)^{x+y}\binom{x+y}{x}\left(\frac{2}{5}\right)^{x}\left(\frac{3}{5}\right)^{y} & x+y\leq 5, x\geq 0, y\geq 0 \\ 0 & \text{otherwise} \end{cases}$$