Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,2.2.1 2.2.9 2.3.10 2.3.11 2.3.12 2.4.3 2.5.10 2.5.11 2.6.5 2.7.7 2.7.8 2.7.9 2.8.10 and 2.9.6

Problem 2.2.1

(a) We wish to find the value of c that makes the PMF sum up to one.

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

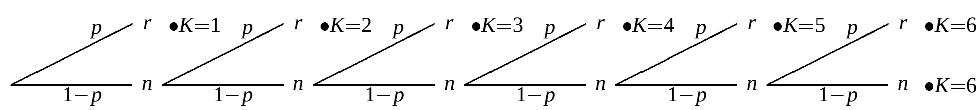
Therefore, $\sum_{n=0}^{2} P_N(n) = c + c/2 + c/4 = 1$, implying $c = 4/7$.

(b) The probability that $N \leq 1$ is

$$P[N \leq 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7$$

Problem 2.2.9

(a) In the setup of a mobile call, the phone will send the "SETUP" message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability p. Of course, the phone stops trying as soon as there is a success. Using r to denote a successful response, and n a non-response, the sample tree is



(b) We can write the PMF of K, the number of "SETUP" messages sent as

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1,2,\ldots,5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases}$$

Note that the expression for $P_K(6)$ is different because $K = 6$ if either there was a success or a failure on the sixth attempt. In fact, $K = 6$ whenever there were failures on the first five attempts which is why $P_K(6)$ simplifies to $(1-p)^5$.

(c) Let B denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is $P[B] = (1-p)^6$. To be sure that $P[B] \leq 0.02$, we need $p \geq 1 - (0.02)^{1/6} = 0.479$.

Problem 2.3.10

(a) Since each day is independent of any other day, $P[W_{33}]$ is just the probability that a winning lottery ticket was bought. Similarly for $P[L_{87}]$ and $P[N_{99}]$ become just the probability that a losing ticket was bought and that no ticket was bought on a single day, respectively. Therefore

$$P[W_{33}] = p/2 \qquad P[L_{87}] = (1-p)/2 \qquad P[N_{99}] = 1/2$$

(b) Supose we say a success occurs on the kth trial if on day k we buy a ticket. Otherwise, a failure occurs. The probability of success is simply $1/2$. The random variable K is just the number of trials until the first success and has the geometric PMF

$$P_K(k) = \begin{cases} (1/2)(1/2)^{k-1} = (1/2)^k & k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(c) The probability that you decide to buy a ticket and it is a losing ticket is $(1-p)/2$, independent of any other day. If we view buying a losing ticket as a Bernoulli success, R, the number of losing lottery tickets bought in m days, has the binomial PMF

$$P_R(r) = \begin{cases} \binom{m}{r}[(1-p)/2]^r[(1+p)/2]^{m-r} & r = 0, 1, \ldots, m \\ 0 & \text{otherwise} \end{cases}$$

(d) Letting D be the day on which the j-th losing ticket is bought, we can find the probability that $D = d$ by noting that $j-1$ losing tickets must have been purchased in the $d-1$ previous days. Therefore D has the Pascal PMF

$$P_D(d) = \begin{cases} \binom{j-1}{d-1}[(1-p)/2]^d[(1+p)/2]^{d-j} & d = j, j+1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.3.11

(a) Let S_n denote the event that the Sixers win the series in n games. Similarly, C_n is the event that the Celtics in in n games. The Sixers win the series in 3 games if they win three straight, which occurs with probability

$$P[S_3] = (1/2)^3 = 1/8$$

The Sixers win the series in 4 games if they win two out of the first three games and they win the fourth game so that

$$P[S_4] = \binom{3}{2}(1/2)^3(1/2) = 3/16$$

The Sixers win the series in five games if they win two out of the first four games and then win game five. Hence,

$$P[S_5] = \binom{4}{2}(1/2)^4(1/2) = 3/16$$

By symmetry, $P[C_n] = P[S_n]$. Further we observe that the series last n games if either the Sixers or the Celtics win the series in n games. Thus,

$$P[N = n] = P[S_n] + P[C_n] = 2P[S_n]$$

Consequently, the total number of games, N, played in a best of 5 series between the Celtics and the Sixers can be described by the PMF

$$P_N(n) = \begin{cases} 2(1/2)^3 = 1/4 & n = 3 \\ 2\binom{3}{1}(1/2)^4 = 3/8 & n = 4 \\ 2\binom{4}{2}(1/2)^5 = 3/8 & n = 5 \\ 0 & \text{otherwise} \end{cases}$$

(b) For the total number of Celtic wins W, we note that if the Celtics get $w < 3$ wins, then the Sixers won the series in $3 + w$ games. Also, the Celtics win 3 games if they win the series in 3,4, or 5 games. Mathematically,

$$P[W = w] = \begin{cases} P[S_{3+w}] & w = 0, 1, 2 \\ P[C_3] + P[C_4] + P[C_5] & w = 3 \end{cases}$$

Thus, the number of wins by the Celtics, W, has the PMF shown below.

$$P_W(w) = \begin{cases} P[S_3] = 1/8 & w = 0 \\ P[S_4] = 3/16 & w = 1 \\ P[S_5] = 3/16 & w = 2 \\ 1/8 + 3/16 + 3/16 = 1/2 & w = 3 \\ 0 & \text{otherwise} \end{cases}$$

(c) The number of Celtic losses L equals the number of Sixers' wins W_S. This implies $P_L(l) = P_{W_S}(l)$. Since either team is equally likely to win any game, by symmetry, $P_{W_S}(w) = P_W(w)$. This implies $P_L(l) = P_{W_S}(l) = P_W(l)$. The complete expression of for the PMF of L is

$$P_L(l) = P_W(l) = \begin{cases} 1/8 & l = 0 \\ 3/16 & l = 1 \\ 3/16 & l = 2 \\ 1/2 & l = 3 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.3.12

be a binomial random variable for n trials and success probability $p = a/(a+b)$. First, we observe that the sum of over all possible values of the PMF of K is

$$\sum_{k=0}^{n} P_K(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k} = \frac{\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}{(a+b)^n}$$

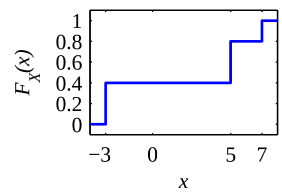
Since $\sum_{k=0}^{n} P_K(k) = 1$, we see that

$$(a+b)^n = (a+b)^n \sum_{k=0}^{n} P_K(k) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Problem 2.4.3

(a) Similar to the previous problem, the graph of the CDF is shown below.

$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \le x < 5 \\ 0.8 & 5 \le x < 7 \\ 1 & x \ge 7 \end{cases}$$



(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.5.10

By the definition of the expected value,

$$\begin{aligned} E[X_n] &= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\ &= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-1-(x-1))!} p^{x-1}(1-p)^{n-1-(x-1)} \end{aligned}$$

With the substitution $x' = x - 1$, we have

$$E[X_n] = np \underbrace{\sum_{x'=0}^{n-1} \binom{n-1}{x'} p^{x'} (1-p)^{n-x'}}_{1} = np \sum_{x'=0}^{n-1} P_{X_{n-1}}(x) = np$$

The above sum is 1 because it is he sum of a binomial random variable for $n-1$ trials over all possible values.

Problem 2.5.11

We write the sum as a double sum in the following way:

$$\sum_{i=0}^{\infty} P[X > i] = \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} P_X(j)$$

At this point, the key step is to reverse the order of summation. You may need to make a sketch of the feasible values for i and j to see how this reversal occurs. In this case,

$$\sum_{i=0}^{\infty} P[X > i] = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} P_X(j) = \sum_{j=1}^{\infty} j P_X(j) = E[X]$$

Problem 2.6.5

(a) The source continues to transmit packets until one is received correctly. Hence, the total number of times that a packet is transmitted is $X = x$ if the first $x-1$ transmissions were in error. Therefore the PMF of X is

$$P_X(x) = \begin{cases} q^{x-1}(1-q) & x = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

(b) The time required to send a packet is a millisecond and the time required to send an acknowledgment back to the source takes another millisecond. Thus, if X transmissions of a packet are needed to send the packet correctly, then the packet is correctly received after $T = 2X-1$ milliseconds. Therefore, for an odd integer $t > 0$, $T = t$ iff $X = (t+1)/2$. Thus,

$$P_T(t) = P_X((t+1)/2) = \begin{cases} q^{(t-1)/2}(1-q) & t = 1,3,5,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.7.7

Let W denote the event that a circuit works. The circuit works and generates revenue of k dollars if all of its 10 constituent devices work. For each implementation, standard or ultra-reliable, let R denote the profit on a device. We can express the expected profit as

$$E[R] = P[W]E[R|W] + P[W^c]E[R|W^c]$$

Let's first consider the case when only standard devices are used. In this case, a circuit works with probability $P[W] = (1-q)^{10}$. The profit made on a working device is $k-10$ dollars while a nonworking circuit has a profit of -10 dollars. That is, $E[R|W] = k-10$ and $E[R|W^c] = -10$. Of course, a negative profit is actually a loss. Using R_s to denote the profit using standard circuits, the expected profit is

$$\begin{aligned} E[R_s] &= (1-q)^{10}(k-10) + (1-(1-q)^{10})(-10) \\ &= (0.9)^{10}k - 10 \end{aligned}$$

And for the ultra-reliable case, the circuit works with probability $P[W] = (1-q/2)^{10}$. The profit per working circuit is $E[R|W] = k-30$ dollars while the profit for a nonworking circuit is $E[R|W^c] = -30$ dollars. The expected profit is

$$\begin{aligned} E[R_u] &= (1-q/2)^{10}(k-30) + (1-(1-q/2)^{10})(-30) \\ &= (0.95)^{10}k - 30 \end{aligned}$$

Now we wish to determine which implementation will generate the most profit. Realizing that both profit functions are linear functions of k, we can plot them versus k and find for which values of k each plan is preferable. The two lines intersect at a value of $k = 80.21$ dollars. So for values of $k <$ \$80.21 using all standard devices results in greater revenue, and for values of $k >$ \$80.21 more revenue will be generated by implementing all ultra-reliable devices. So we can see that when the price commanded for each working circuit is sufficiently high it is worthwhile to spend the extra money to ensure that more working circuits can be produced.

Problem 2.7.8

(a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9,366,819} \approx 1.07 \times 10^{-7}$$

(b) Assuming each ticket is chosen randomly, each of the $2n-1$ other tickets is independently a winner with probability q. The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1-q)^{2n-1-k} & k = 0, 1, \ldots, 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Since there are $K_n + 1$ winning tickets in all, the value of your winning ticket is $W_n = n/(K_n + 1)$ which has mean

$$E[W_n] = nE\left[\frac{1}{K_n+1}\right]$$

Calculating the expected value

$$E\left[\frac{1}{K_n+1}\right] = \sum_{k=0}^{2n-1} \left(\frac{1}{k+1}\right) P_{K_n}(k)$$

is fairly complicated. The trick is to express the sum in terms of the sum of a binomial PMF.

$$\begin{aligned} E\left[\frac{1}{K_n+1}\right] &= \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)} \end{aligned}$$

By factoring out $1/q$, we obtain

$$\begin{aligned} E\left[\frac{1}{K_n+1}\right] &= \frac{1}{2nq} \sum_{k=0}^{2n-1} \binom{2n}{k+1} q^{k+1} (1-q)^{2n-(k+1)} \\ &= \frac{1}{2nq} \underbrace{\sum_{j=1}^{2n} \binom{2n}{j} q^j (1-q)^{2n-j}}_{A} \end{aligned}$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j = 0$. Thus

$$A = 1 - \binom{2n}{0} q^0 (1-q)^{2n-0} = 1 - (1-q)^{2n}$$

This implies

$$E\left[\frac{1}{K_n+1}\right] = \frac{1 - (1-q)^{2n}}{2nq}$$

Our expected return on a winning ticket is

$$E[W_n] = nE\left[\frac{1}{K_n+1}\right] = \frac{1-(1-q)^{2n}}{2q}$$

Note that when $nq \ll 1$, we can use the approximation that $(1-q)^{2n} \approx 1-2nq$ to show that

$$E[W_n] \approx \frac{1-(1-2nq)}{2q} = n \qquad (nq \ll 1)$$

However, in the limit as the value of the prize n approaches infinity, we have

$$\lim_{n\to\infty} E[W_n] = \frac{1}{2q} \approx 4.683\times 10^6$$

That is, as the pot grows to infinity, the expected return on a winning ticket doesn't approach infinity because there is a corresponding increase in the number of other winning tickets. If it's not clear how large n must be for this effect to be seen, consider the following table:

n	10^6	10^7	10^8
$E[W_n]$	9.00×10^5	4.13×10^6	4.68×10^6

When the pot is $1 million, our expected return is $900,000. However, we see that when the pot reaches $100 million, our expected return is very close to $1/(2q)$, less than $5 million!

Problem 2.7.9

(a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9,366,819} \approx 1.07\times 10^{-7}$$

(b) Assuming each ticket is chosen randomly, each of the $2n-1$ other tickets is independently a winner with probability q. The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1-q)^{2n-1-k} & k = 0,1,\ldots,2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Since the pot has $n+r$ dollars, the expected amount that you win on your ticket is

$$E[V] = 0(1-q) + qE\left[\frac{n+r}{K_n+1}\right] = q(n+r)E\left[\frac{1}{K_n+1}\right]$$

Note that $E[1/K_n+1]$ was also evaluated in Problem 2.7.8. For completeness, we repeat those steps here.

$$\begin{aligned} E\left[\frac{1}{K_n+1}\right] &= \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \\ &= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)} \end{aligned}$$

By factoring out $1/q$, we obtain

$$E\left[\frac{1}{K_n+1}\right] = \frac{1}{2nq}\sum_{k=0}^{2n-1}\binom{2n}{k+1}q^{k+1}(1-q)^{2n-(k+1)}$$
$$= \frac{1}{2nq}\underbrace{\sum_{j=1}^{2n}\binom{2n}{j}q^j(1-q)^{2n-j}}_{A}$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j=0$. Thus $A = 1-\binom{2n}{0}q^0(1-q)^{2n-0}$, which implies

$$E\left[\frac{1}{K_n+1}\right] = \frac{A}{2nq} = \frac{1-(1-q)^{2n}}{2nq}$$

The expected value of your ticket is

$$E[V] = \frac{q(n+r)[1-(1-q)^{2n}]}{2nq} = \frac{1}{2}\left(1+\frac{r}{n}\right)[1-(1-q)^{2n}]$$

Each ticket tends to be more valuable when the carryover pot r is large and the number of new tickets sold, $2n$, is small. For any fixed number n, corresponding to $2n$ tickets sold, a sufficiently large pot r will guarantee that $E[V]>1$. For example if $n=10^7$, (20 million tickets sold) then

$$E[V] = 0.44\left(1+\frac{r}{10^7}\right)$$

If the carryover pot r is 30 million dollars, then $E[V]=1.76$. This suggests that buying a one dollar ticket is a good idea. This is an unusual situation because normally a carryover pot of 30 million dollars will result in far more than 20 million tickets being sold.

(c) So that we can use the results of the previous part, suppose there were $2n-1$ tickets sold before you must make your decision. If you buy one of each possible ticket, you are guaranteed to have one winning ticket. From the other $2n-1$ tickets, there will be K_n winners. The total number of winning tickets will be K_n+1. In the previous part we found that

$$E\left[\frac{1}{K_n+1}\right] = \frac{1-(1-q)^{2n}}{2nq}$$

Let R denote the expected return from buying one of each possible ticket. The pot had r dollars beforehand. The $2n-1$ other tickets are sold add $n-1/2$ dollars to the pot. Furthermore, you must buy $1/q$ tickets, adding $1/(2q)$ dollars to the pot. Since the cost of the tickets is $1/q$ dollars, your expected profit

$$E[R] = E\left[\frac{r+n-1/2+1/(2q)}{K_n+1}\right] - \frac{1}{q}$$
$$= \frac{q(2r+2n-1)+1}{2q}E\left[\frac{1}{K_n+1}\right] - \frac{1}{q}$$
$$= \frac{[q(2r+2n-1)+1](1-(1-q)^{2n})}{4nq^2} - \frac{1}{q}$$

For fixed n, sufficiently large r will make $E[R] > 0$. On the other hand, for fixed r, $\lim_{n\to\infty} E[R] = -1/(2q)$. That is, as n approaches infinity, your expected loss will be quite large.

Problem 2.8.10

We wish to minimize the function

$$e(\hat{x}) = E\left[(X-\hat{x})^2\right]$$

with respect to $\hat{x}$. We can expand the square and take the expectation while treating $\hat{x}$ as a constant. This yields

$$e(\hat{x}) = E\left[X^2 - 2\hat{x}X + \hat{x}^2\right] = E\left[X^2\right] - 2\hat{x}E[X] + \hat{x}^2$$

Solving for the value of $\hat{x}$ that makes the derivative $de(\hat{x})/d\hat{x}$ equal to zero results in the value of $\hat{x}$ that minimizes $e(\hat{x})$. Note that when we take the derivative with respect to $\hat{x}$, both $E\left[X^2\right]$ and $E[X]$ are simply constants.

$$\frac{d}{d\hat{x}}\left(E\left[X^2\right] - 2\hat{x}E[X] + \hat{x}^2\right) = 2E[X] - 2\hat{x} = 0$$

Hence we see that $\hat{x} = E[X]$. In the sense of mean squared error, the best guess for a random variable is the mean value. In Chapter 9 this idea is extended to develop minimum mean squared error estimation.

Problem 2.9.6

(a) Consider each circuit test as a binomial trial such that a failed circuit is called a success. The number of trias until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(b) The probability there are at least 20 tests is

$$P[B] = P[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19}$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P[B]} & n \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(c) Given the event B the conditional expectation of N is

$$E[N|B] = \sum_n nP_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20}p$$

Making the substitution $j = n - 19$ yields

$$E[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1} p = 1/p + 19$$

We see that in the above sum, we effectively have the expected value of $J+19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.