Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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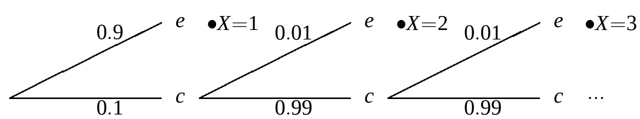
Problem Solutions : Yates and Goodman,11.1.3 11.1.4 11.2.5 11.2.6 11.3.3 11.3.4 11.4.2 11.6.2 11.7.3 11.8.3 11.9.1 11.10.1 and 11.11.7

Problem 11.1.3

the occurrences of packets in error. It would seem that $N(t)$ cannot be a renewal process because the interarrival times seem to depend on the previous interarrival times. However, following a packet error, the sequence of packets that are correct (c) or in error (e) up to and including the next error is given by the tree



Assuming that sending a packet takes one unit of time, the time X until the next packet error has the PMF

$$P_X(x) = \begin{cases} 0.9 & x = 1 \\ 0.001(0.99)^{x-2} & x = 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

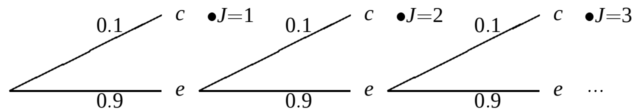
Thus, following an error, the time until the next error always has the same PMF. Moreover, this time is independent of previous interarrival times since it depends only on the Bernoulli trials following a packet error. It would appear that $N(t)$ is a renewal process; however, there is one additional complication. At time 0, we need to know the probability p of an error for the first packet. If $p = 0.9$, then X_1, the time until the first error, has the same PMF as X above and the process is a renewal process. If $p \neq 0.9$, then the time until the first error is different from subsequent renewal times. In this case, the process is a delayed renewal process.

Problem 11.1.4

tricky. Just as in the solution to Problem 11.1.3, its unclear in what mode the system starts at time 0. For the moment, we ignore this problem and consider what happens immediately following an arrival, that is, the arrival of a packet in error following a correct packet. In the following diagram, we will use c and e to denote correct packets and error packets while E will mark the arrivals of a first packet in error following a correct packet. The basic sequence we will observe resembles

$$\cdots cccE \underbrace{ee\cdots e}_{J-1 \text{ err}} c \underbrace{cc\cdots c}_{K-1 \text{ ok}} E \underbrace{ee\cdots e}_{J-1 \text{ err}} c \underbrace{cc\cdots c}_{K-1 \text{ ok}} E \cdots$$

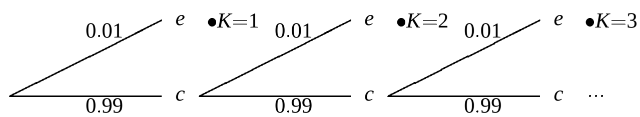
Following the arrival of a packet in error, we will need to observe J packets to see the first correct packet and then we will need to see K additional packets to see the first error packet following a correct packet. The PMF of J can be deduced from the following tree which shows the sequence of packets that are correct (c) or in error (e) up to and including the first correct packet.



Assuming that sending a packet takes one unit of time, the time J until the first correct packet has the PMF

$$P_X(x) = \begin{cases} (0.9)^{j-1}(0.1) & j = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Following a correct packet, we will observe K packets to see the next error. This is shown in the following tree:



The corresponding PMF of K is

$$P_K(k) = \begin{cases} (0.99)^{k-1}(0.01) & k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The time between arrivals is the random variable

$$X' = J + K$$

Following an arrival, the number of packets until the next arrival is always a random variable X'. Moreover, J and K are independent of the packet errors that occurred up to the previous arrival. Thus $N'(t)$ is at the very least a delayed renewal process. Furthermore, if we know that at time $t = 1$, the error probability of the first packet is 0.9, then we know that the $N'(t)$ process is in fact a renewal process.

Although finding the exact PMF of X' is not very difficult, note that we do not even need to find it to make the above argument.

Problem 11.2.5

We start with the case when $t \geq 2$. When each service time is equally likely to be either 1 minute or 2 minutes, we have the following situation. Let M_1 denote those customers that arrived in the interval $(t-1, 1]$. All M_1 of these customers will be in the bank at time t and M_1 is a Poisson random variable with mean λ.

Let M_2 denote the number of customers that arrived during $(t-2, t-1]$. Of course, M_2 is Poisson with expected value λ. We can view each of the M_2 customers as flipping a coin to determine whether to choose a 1 minute or a 2 minute service time. Only those customers that chooses a 2 minute service time will be in service at time t. Let M_2' denote those customers choosing a 2 minute service time. It should be clear that M_2' is a Poisson number of Bernoulli random variables. Theorem 11.5 verifies that using Bernoulli trials to decide whether the arrivals of a rate λ Poisson process should be counted yields a Poisson process of rate $p\lambda$. A consequence of this result is that a Poisson number of Bernoulli (success probability p) random variables has Poisson PMF with mean $p\lambda$. In this case, M_2' is Poisson with mean $\lambda/2$. Moreover, the number of customers in service at time t is $N(t) = M_1 + M_2'$. Since M_1 and M_2' are independent Poisson random variables, their sum $N(t)$ also has a Poisson PMF. This was verified in Example 7.12. Hence $N(t)$ is Poisson with mean $E[N(t)] = E[M_1] + E[M_2'] = 3\lambda/2$. The PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (3\lambda/2)^n e^{-3\lambda/2}/n! & n = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \qquad (t \geq 2)$$

Now we can consider the special cases arising when $t < 2$. When $0 \leq t < 1$, every arrival is still in service. Thus the number in service $N(t)$ equals the number of arrivals and has the PMF

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq t \leq 1)$$

When $1 \leq t < 2$, let M_1 denote the number of customers in the interval $(t-1,t]$. All M_1 customers arriving in that interval will be in service at time t. The M_2 customers arriving in the interval $(0,t-1]$ must each flip a coin to decide one a 1 minute or two minute service time. Only those customers choosing the two minute service time will be in service at time t. Since M_2 has a Poisson PMF with mean $\lambda(t-1)$, the number M_2' of those customers in the system at time t has a Poisson PMF with mean $\lambda(t-1)/2$. Finally, the number of customers in service at time t has a Poisson PMF with mean $E[N(t)] = E[M_1] + E[M_2'] = \lambda + \lambda(t-1)/2$. Hence, the PMF of $N(t)$ becomes

$$P_{N(t)}(n) = \begin{cases} (\lambda(t+1)/2)^n e^{-\lambda(t+1)/2}/n! & n = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \qquad (1 \leq t \leq 2)$$

Problem 11.2.6

process is not a renewal process. However, before we go into the details, it is perhaps better to explain why. Suppose $\lambda_0 \gg \lambda_1$. In this case, type 0 arrivals occur much more frequently than type 1 arrivals. As a consequence, we are likely to see a sample path with alternating short and long interarrival times. If we have observed a sequence of interarrival times that are short, long, short, long, then it becomes possible for us to guess whether the next interarrival time is short or long. Hence, we can use the previous interarrival times to deduce something about the current interarrival time. This would suggest that the interarrival times are dependent which would imply the alternating process is neither a renewal process nor a Poisson process. Verifying this intuition is complicated because the first arrival can be of either type. For $i = 0,1$, let A_i denote the event that the first arrival is of type i. Since the two processes are Poisson processes, Theorem 11.6 says that

$$P[A_0] = \frac{\lambda_0}{\lambda_0 + \lambda_1} \qquad P[A_1] = \frac{\lambda_1}{\lambda_0 + \lambda_1}$$

In addition, given we learn the type of the first arrival, then subsequent interarrival times are conditionally independent because we know in each interval what type of arrival that we are looking for (type 0 or type 1) and because of the memorylessness of each $N_i(t)$ process. In particular

- If A_0 occurs and the first arrival is type 0, then X_2 will be exponential with parameter λ_1, X_3 will be exponential with parameter λ_0, X_4 will be exponential with parameter λ_1, and so on. The conditional joint PDF of the first three arrivals will be

$$\begin{aligned} f_{X_1,X_2,X_3|A_0}(x_1,x_2,x_3) &= f_{X_1|A_0}(x_1)\, f_{X_2|A_0}(x_2)\, f_{X_3|A_0}(x_3) \\ &= f_{X_1|A_0}(x_1)\, \lambda_1 e^{-\lambda_1 x_2} \lambda_0 e^{-\lambda_0 x_3} \end{aligned}$$

- If A_1 occurs and the first arrival is type 1, then X_2 will be exponential with parameter λ_0, X_3 will be exponential with parameter λ_1, X_4 will be exponential with parameter λ_0, and so on. The conditional joint PDF of the first three arrivals will be

$$\begin{aligned} f_{X_1,X_2,X_3|A_1}(x_1,x_2,x_3) &= f_{X_1|A_1}(x_1)\, f_{X_2|A_1}(x_2)\, f_{X_3|A_1}(x_3) \\ &= f_{X_1|A_1}(x_1)\, \lambda_0 e^{-\lambda_0 x_2} \lambda_1 e^{-\lambda_1 x_3} \end{aligned}$$

To finish the verification that X_1, X_2, and X_3 are dependent, we must find the conditional PDFs $f_{X_1|A_0}(x_1)$ and $f_{X_1|A_1}(x_1)$. For $i = 0, 1$, let Y_i denote the arrival time of the first type i arrival. Since each process $N_i(t)$ is Poisson, for $x \geq 0$,

$$P[Y_i > x] = e^{-\lambda_i x}$$

In addition, note that $X_1 = \min(Y_0, Y_1)$ and that $A_0 = \{Y_0 < Y_1\}$. This implies

$$\begin{aligned} F_{X_1|A_0}(x_1) &= 1 - P[X_1 > x_1 | A_0] \\ &= 1 - P[\min(Y_0,Y_1) > x_1, A_0]/P[A_0] \\ &= 1 - P[Y_0 > x_1, Y_1 > x_1, Y_0 < Y_1]/P[A_0] \\ &= 1 - P[x_1 < Y_0 < Y_1]/P[A_0] \end{aligned}$$

Thus we need to calculate

$$\begin{aligned} P[x_1 < Y_0 < Y_1] &= \int_{x_1}^{\infty} \int_{y_0}^{\infty} f_{Y_0}(y_0)\, f_{Y_1}(y_1)\, dy_1\, dy_0 \\ &= \int_{x_1}^{\infty} \lambda_0 e^{-\lambda_0 y_0} \int_{y_0}^{\infty} \lambda_1 e^{-\lambda_1 y_1}\, dy_1\, dy_0 \\ &= \int_{x_1}^{\infty} \lambda_0 e^{-(\lambda_0+\lambda_1) y_0}\, dy_0 \\ &= \frac{\lambda_0}{\lambda_0+\lambda_1} e^{-(\lambda_0+\lambda_1)x_1} = P[A_0] e^{-(\lambda_0+\lambda_1)x_1} \end{aligned}$$

This implies

$$F_{X_1|A_0}(x_1) = 1 - e^{-(\lambda_0+\lambda_1)x_1}$$

The same approach can be used to show that

$$F_{X_1|A_1}(x_1) = 1 - e^{-(\lambda_0+\lambda_1)x_1}$$

In short, the time of the first arrival is independent of whether that arrival is type 0 or type 1. Although this may seem surprising, it makes sense when we consider the events $X_1 > x_1$ and the event A_0. Given we observe that $X_1 > x_1$, we know that neither $N_0(t)$ nor $N_1(t)$ had an arrival by time x_1. Given we know this fact, the memoryless property of the Poisson process tells us that the probability the type 0 arrival will occur first is still $P[A_0]$, independent of how long we have already waited. Finally, we see that

$$F_{X_1}(x_1) = F_{X_1|A_0}(x_1)\,P[A_0] + F_{X_1|A_1}(x_1)\,P[A_1] = 1 - e^{-(\lambda_0+\lambda_1)x_1}$$

That is, $F_{X_1|A_0}(x_1) = F_{X_1|A_1}(x_1) = F_{X_1}(x_1)$. Furthermore, by taking derivatives, we have

$$f_{X_1}(x_1) = f_{X_1|A_0}(x_1) = f_{X_1|A_1}(x_1) = (\lambda_0+\lambda_1)e^{-(\lambda_0+\lambda_1)x_1}$$

Finally, we can evaluate the joint PDF of X_1, X_2, and X_3.

$$\begin{aligned} f_{X_1,X_2,X_2}(x_1,x_2,x_2) &= f_{X_1,X_2,X_3|A_0}(x_1,x_2,x_3)\,P[A_0] + f_{X_1,X_2,X_3|A_1}(x_1,x_2,x_3)\,P[A_1] \\ &= f_{X_1|A_0}(x_1)\, f_{X_2|A_0}(x_2)\, f_{X_3|A_0}(x_3)\,P[A_0] \\ &\quad + f_{X_1|A_1}(x_1)\, f_{X_2|A_1}(x_2)\, f_{X_3|A_1}(x_3)\,P[A_1] \end{aligned}$$

For x_1, x_2, x_3 nonnegative, substituting the various conditional PDFs yields

$$f_{X_1,X_2,X_2}(x_1,x_2,x_2) = \lambda_0\lambda_1 e^{-(\lambda_0+\lambda_1)x_1}\left(\lambda_0 e^{-\lambda_1 x_2 - \lambda_0 x_3} + \lambda_1 e^{-\lambda_0 x_2 - \lambda_1 x_3}\right)$$

We see that the joint PDF $f_{X_1,X_2,X_3}(x_1,x_2,x_3)$ does not factor into a product of marginal PDFs. Hence the interarrival times $X_0, X_1, \ldots$ are not independent and the $N(t)$ process is not a renewal process.

It is possible to identify an embedded renewal process. Let $N'(t)$ denote the even numbered arrivals of $N(t)$. Also, let $X_1', X_2', \ldots$ denote interarrival times of the $N'(t)$ process. If Y_i denotes an arbitrary interarrival process of the $N_i(t)$ process, then some thought will make the following observations clear.

- The first arrival of the $N'(t)$ process occurs exactly when each process $N_i(t)$ has had an arrival. Thus,

  $$X_1' = \max(Y_0, Y_1)$$

- After the first arrival of the $N'(t)$ process, we must wait for either

  - One arrival of the $N_0(t)$ process followed by one arrival of the $N_1(t)$ process
  - One arrival of the $N_1(t)$ process followed by one arrival of the $N_0(t)$ process

  The order in which we wait for these arrivals will depend on whether the very first arrival was type 0 or type 1. Nevertheless, for $n > 1$,

  $$X_n' = Y_0 + Y_1$$

  That is, $X_2', X_3', \ldots$ are identical random variables.

- Because $N_0(t)$ and $N_1(t)$ are Poisson processes, the interarrival times $X_1' X_2', \ldots$ are independent random variables

From these facts, we can conclude that $N'(t)$ is a delayed renewal process.

Problem 11.3.3

(a) We can define a renewal process such that a renewal occurs whenever we produce an intermediate digit. Let $Y_1, Y_2, \ldots$ denote the inter-renewal times. Note that Y has a truncated geometric PMF. For this problem, its easier to work with the complementary CDF of Y. In particular,

$$P[Y > j] = \begin{cases} 1 & j < 0 \\ q^j & j = 0, 1, \ldots, 7 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of Y is

$$E[Y] = \sum_{j=0}^{\infty} P[Y > j] = \sum_{j=0}^{7} q^j = \frac{1-q^8}{1-q}$$

Since one intermediate digit is produced in each renewal period,

$$\alpha = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[Y]} = \frac{1-q}{1-q^8}$$

(b) The value of each intermediate digit produced can be viewed as an independent trial. Let R_k denote the number of code bits produced by the kth intermediate digit. Note that $R = 1$ if intermediate digit 8 is produced; otherwise, $R = 4$. Since the probability that intermediate digit 8 is produced is q^8,

$$\beta = E[R] = 4(1-q^8) + q^8 = 4 - 3q^8$$

Note that

$$\hat{M}(k) = R_1 + \cdots + R_k$$

Since $R_1, R_2, \ldots$ is an iid random sequence, the strong law of large number says that

$$\lim_{k\to\infty} \frac{\hat{M}(k)}{k} = E[R] = 4 - 3q^8 \qquad \text{w.p. } 1$$

(c) Now consider a renewal reward process in which renewals occur whenever an intermediate digit is produced and the reward equals the number of code bits produced. That is, the sequence of (Y_k, R_k) are a renewal reward process. The longterm rate of code bits per unit time is

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{E[R]}{E[Y]} = \alpha\beta = \frac{(1-q)(4-3q^8)}{1-q^8}$$

(d) The RLL coding compresses the bit stream if $\lim_{t\to\infty} M(t)/t < 1$. This requires that

$$4 - 4q - 2q^8 + 3q^9 < 1$$

Plotting the left side function of q shows that $q > 0.777$ ensures that RLL coding compresses the sequence.

Problem 11.3.4

(a) We can define a renewal reward process such that a renewal occurs whenever the source produces a 1. Over a renewal period, the reward R equals the number of intermediate digits produced. The time X between renewals is equal to the number of source bits produced during the renewal period. Note that $X > j$ iff the first j source bits after a renewal are all zero. Hence,

$$P[X > j] = \begin{cases} 1 & j = 0 \\ pq^{j-1} & j = 1, 2, \ldots \end{cases}$$

and

$$E[X] = \sum_{j=0}^{\infty} P[X > j] = 1 + \frac{p}{1-q}$$

By similar reasoning, $R > r$ iff the the first $8r$ source bits are all zero. For example, $R > 1$, if the first 8 digits after the renewal are all zero.

$$P[R > r] = \begin{cases} 1 & r = 0 \\ pq^{8r-1} & r = 1, 2, \ldots \end{cases}$$

This implies

$$E[R] = \sum_{r=0}^{\infty} P[R > r] = 1 + \frac{pq^7}{1-q^8}$$

By the renewal reward theorem,

$$\alpha = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{E[R]}{E[X]} = \frac{(1-q)(1-q^8+pq^7)}{(1-q^8)(1-q+p)}$$

(b) Consider a renewal reward process in which one unit of time is required to produce one intermediate digit. Renewals occur whenever one of the intermediate digits $0, 1, \ldots, 7$ is produced. Hence, a renewal period produces a sequence of intermediate digits $88\cdots 8x$ where $x \in \{0, 1, \ldots, 7\}$. In this case, the expected time between renewals equals $E[R]$, as defined in part (a). Over a renewal period, the reward C equals the number of code bits produced. Note that the first $R-1$ intermediate digits produce $(R-1)$ code bits. For the final intermediate digit, 4 code bits are produced. Hence $C = R+3$. Moreover,

$$\beta = \lim_{k\to\infty} \frac{\hat{M}(k)}{k} = \frac{E[C]}{E[R]} = \frac{E[R]+3}{E[R]}$$

(c) As in part (a), we consider each time a 1 is produced by the source to be a renewal. The time between renewals X is as defined in part (a). We define a reward as the number of code bits C' produced during a renewal period. As explained in part (b), if R intermediate digits are produced, then $C' = R+3$ code bits are produced. In this case, the renewal reward theorem implies

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{E[C']}{E[X]} = \frac{E[R]+3}{E[X]} = \alpha\beta$$

(d) The short answer is that the procedure compresses the source sequence when

$$\lim_{t\to\infty} M(t)/t = \alpha\beta < 1$$

This compression is significant when long sequences of zeroes are typical. This occur when p and q are close to 1. This answer is a little too simple because it doesn't account for the fact even if p is not close to 1, a value of q that is very close to 1 can result in compression. The condition $\alpha\beta < 1$ is the same as $E[R]+3 < E[X]$. In terms of p and q, we have

$$4 + \frac{pq^7}{1-q^8} < 1 + \frac{p}{1-q}$$

This can be simplified to

$$p > \frac{3(1-q^8)(1-q)}{1-q^7} = 3(1-q)h(q)$$

where $h(q) = (1-q^8)/(1-q^7)$. We will show that $h(q)$ is an increasing function such that $1 \le h(q) \le 8/7$ for $0 \le q \le 1$. Note that we can write $h(q)$ as

$$h(q) = 1 + \frac{(1-q)q^7}{1-q^7}$$

Since $1-q^7 = (1-q)(1+q+q^2+\cdots+q^6)$, we can write

$$h(q) = 1 + \frac{q^7}{1+q+q^2+\cdots+q^6} = 1 + \frac{1}{\sum_{i=1}^{7} 1/q^i}$$

Since $1/q^i$ is a decreasing function in q for all $i \ge 1$, $\sum_{i=1}^{7} 1/q^i$ is a decreasing function and $h(q)$ is an increasing function. Thus, for $0 \le q \le 1$,

$$1 = h(0) \le h(q) \le h(1) = 8/7$$

Thus a necesssary condition for compression is $p > 3(1-q)$. A sufficient condition for compression is $p > 24(1-q)/7$. This implies that if $q \le 2/3$ then we cannot have any compression, no matter what the value of p.

Problem 11.4.2

fact by fact to identify the information given.

- "… each read or write operation reads or writes an entire file and that files contain a geometric number of sectors with mean 50."

  This statement says that the length L of a file has PMF

$$P_L(l) = \begin{cases} (1-p)^{l-1}p & l = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

  with $p = 1/50 = 0.02$. This says that when we write a sector, we will write another sector with probability $49/50 = 0.98$. In terms of our Markov chain, if we are in the write state, we write another sector and stay in the write state with probability $P_{22} = 0.98$. This fact also implies $P_{20} + P_{21} = 0.02$.

Also, since files that are read obey the same length distribution,

$$P_{11} = 0.98 \qquad P_{10} + P_{12} = 0.02$$

- "Further, suppose idle periods last for a geometric time with mean 500."

  This statement simply says that given the system is idle, it remains idle for another unit of time with probability $P_{00} = 499/500 = 0.998$. This also says that $P_{01} + P_{02} = 0.002$.

- "After an idle period, the system is equally likely to read or write a file."

  Given that at time n, $X_n = 0$, this statement says that the conditional probability that

$$P[X_{n+1} = 1 | X_n = 0, X_{n+1} \neq 0] = \frac{P_{01}}{P_{01} + P_{02}} = 0.5$$

  Combined with the earlier fact that $P_{01} + P_{02} = 0.002$, we learn that

$$P_{01} = P_{02} = 0.001$$

- "Following the completion of a read, a write follows with probability 0.8."

  Here we learn that given that at time n, $X_n = 1$, the conditional probability that

$$P[X_{n+1} = 2 | X_n = 1, X_{n+1} \neq 1] = \frac{P_{12}}{P_{10} + P_{12}} = 0.8$$

  Combined with the earlier fact that $P_{10} + P_{12} = 0.02$, we learn that

$$P_{10} = 0.004 \qquad P_{12} = 0.016$$

- "However, on completion of a write operation, a read operation follows with probability 0.6."

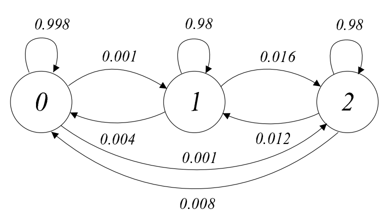
  Now we find that given that at time n, $X_n = 2$, the conditional probability that

$$P[X_{n+1} = 1 | X_n = 2, X_{n+1} \neq 2] = \frac{P_{21}}{P_{20} + P_{21}} = 0.6$$

  Combined with the earlier fact that $P_{20} + P_{21} = 0.02$, we learn that

$$P_{20} = 0.008 \qquad P_{21} = 0.012$$

The complete tree is



Problem 11.6.2
state probabilities. By Theorem 11.10, the probability of state k at time n is

$$p_k(n) = \sum_{i=0}^{\infty} p_i(n-1)P_{ik}$$

Since $P_{ik} = q$ for every state i,

$$p_k(n) = q\sum_{i=0}^{\infty} p_i(n-1) = q$$

Thus for any time $n > 0$, the probability of state k is q.

Problem 11.7.3
states j and i communicate, then sometimes when we go from state j back to state j, we will pass through state i. If $E\left[T_{ij}\right] = \infty$, then on those occasions we pass through i, the expectred time to go to back to j will be infinite. This would suggest $E\left[T_{jj}\right] = \infty$ and thus state j would not be positive recurrent. Using a math to prove this requires a little bit of care.

Suppose $E\left[T_{ij}\right] = \infty$. Since i and j communicate, we can find n, the smallest nonnegative integer such that $P_{ji}^{(n)} > 0$. Given we start in state j, let G_i denote the event that we go through state i on our way back to j. By conditioning on G_j,

$$E\left[T_{jj}\right] = E\left[T_{jj}|G_i\right]P[G_i] + E\left[T_{jj}|G_i^c\right]P[G_i^c]$$

Since $E\left[T_{jj}|G_i^c\right]P[G_i^c] \geq 0$,

$$E\left[T_{jj}\right] \geq E\left[T_{jj}|G_i\right]P[G_i]$$

Given the event G_i, $T_{jj} = T_{ji} + T_{ij}$. This implies

$$E\left[T_{jj}|G_i\right] = E\left[T_{ji}|G_i\right] + E\left[T_{ij}|G_i\right] \geq E\left[T_{ij}|G_i\right]$$

Since the random variable T_{ij} assumes that we start in state i, $E\left[T_{ij}|G_i\right] = E\left[T_{ij}\right]$. Thus $E\left[T_{jj}|G_i\right] \geq E\left[T_{ij}\right]$. In addition, $P[G_i] \geq P_{ji}^{(n)}$ since there may be paths with more than n hops that take the system from state j to i. These facts imply

$$E\left[T_{jj}\right] \geq E\left[T_{jj}|G_i\right]P[G_i] \geq E\left[T_{ij}\right]P_{ji}^{(n)} = \infty$$

Thus, state j is not positive recurrent, which is a contradiction. Hence, it must be that $E\left[T_{ij}\right] < \infty$.

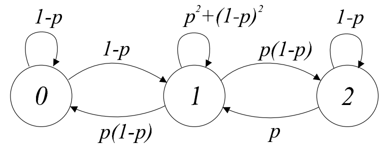
Problem 11.8.3

0 front teller busy, rear teller idle

1 front teller busy, rear teller busy

2 front teller idle, rear teller busy

We will assume the units of time are seconds. Thus, if a teller is busy one second, the teller will become idle in th next second with probability $p = 1/120$. The Markov chain for this system is



We can solve this chain very easily for the stationary probability vector π. In particular,

$$\pi_0 = (1-p)\pi_0 + p(1-p)\pi_1$$

This implies that $\pi_0 = (1-p)\pi_1$. Similarly,

$$\pi_2 = (1-p)\pi_2 + p(1-p)\pi_1$$

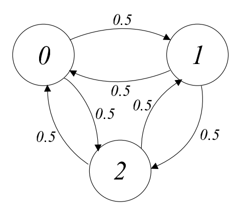
yields $\pi_2 = (1-p)\pi_1$. Hence, by applying $\pi_0 + \pi_1 + \pi_2 = 1$, we obtain

$$\pi_0 = \pi_2 = \frac{1-p}{3-2p} = 119/358$$
$$\pi_1 = \frac{1}{3-2p} = 120/358$$

The stationary probability that both tellers are busy is $\pi_1 = 120/358$.

Problem 11.9.1

Equivalently, we can prove that if $P_{ii} \neq 0$ for some i, then the chain cannot be periodic. So, suppose for state i, $P_{ii} > 0$. Since $P_{ii} = P_{ii}^{(1)}$, we see that the largest d that divides n for all n such that $P_{ii}^{(n)} > 0$ is $d = 1$. Hence, state i is aperiodic and thus the chain is aperiodic.
The converse that $P_{ii} = 0$ for all i implies the chain is periodic is false. As a counterexample, consider the simple chain on the right with $P_{ii} = 0$ for each i. Note that $P_{00}^{(2)} > 0$ and $P_{00}^{(3)} > 0$. The largest d that divides both 2 and 3 is $d = 1$. Hence, state 0 is aperiodic. Since the chain has one communicating class, the chain is also aperiodic.

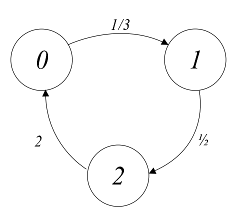


Problem 11.10.1

in each state i, the tiger spends an exponential time with parameter λ_i. When we measure time in hours,

$$\lambda_0 = q_{01} = 1/3 \qquad \lambda_1 = q_{12} = 1/2 \qquad \lambda_2 = q_{20} = 2$$

The corresponding continous time Markov chain is shown below:



The state probabilities satisfy

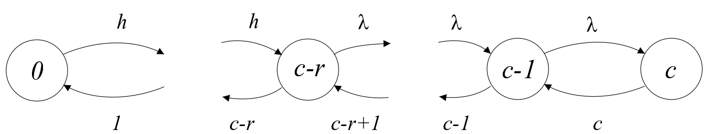
$$\frac{1}{3}p_0 = 2p_2 \qquad \frac{1}{2}p_1 = \frac{1}{3}p_0 \qquad p_0 + p_1 + p_2 = 1$$

The solution is

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 6/11 & 4/11 & 1/11 \end{bmatrix}$$

Problem 11.11.7

Since both types of calls have exponential holding times, the number of calls in the system can be used as the system state. The corresponding Markov chain is



When the number of calls, n, is less than $c-r$, we admit either type of call and $q_{n,n+1} = \lambda + h$. When $n \geq c-r$, we block the new calls and we admit only handoff calls so that $q_{n,n+1} = h$. Since the service times are exponential with an average time of 1 minute, the call departure rate in state n is n calls per minute. Theorem 11.29 says that the stationary probabilities p_n satisfy

$$p_n = \begin{cases} \dfrac{\lambda + h}{n} p_{n-1} & n = 1, 2, \ldots, c-r \\ \dfrac{\lambda}{n} p_{n-1} & n = c-r+1, c-r+2, \ldots, c \end{cases}$$

This implies

$$p_n = \begin{cases} \dfrac{(\lambda + h)^n}{n!} p_0 & n = 1, 2, \ldots, c-r \\ \dfrac{(\lambda + h)^{c-r} \lambda^{n-(c-r)}}{n!} p_0 & n = c-r+1, c-r+2, \ldots, c \end{cases}$$

The requirement that $\sum_{n=1}^{c} p_n = 1$ yields

$$p_0 \left[\sum_{n=0}^{c-r} \frac{(\lambda + h)^n}{n!} + (\lambda + h)^{c-r} \sum_{n=c-r+1}^{c} \frac{\lambda^{n-(c-r)}}{n!} \right] = 1$$

Finally, a handoff call is dropped if and only if a new call finds the system with c calls in progress. The probability that a handoff call is dropped is

$$P[H] = p_c = \frac{(\lambda + h)^{c-r} \lambda^r}{c!} p_0 = \frac{(\lambda + h)^{c-r} \lambda^r / c!}{\sum_{n=0}^{c-r} \frac{(\lambda+h)^n}{n!} + \left(\frac{\lambda + h}{\lambda} \right)^{c-r} \sum_{n=c-r+1}^{c} \frac{\lambda^n}{n!}}$$