Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,10.1.3 10.2.4 10.3.3 10.4.1 10.5.3 10.5.4 10.5.5 and 10.6.4

Problem 10.1.3

By Theorem 10.1, the mean of the output is

$$\begin{aligned}
\mu_Y &= \mu_X \int_{-\infty}^{\infty} h(t)\, dt \\
&= 4\int_0^{\infty} e^{-t/a}\, dt \\
&= -4ae^{-t/a}\Big|_0^{\infty} \\
&= 4a
\end{aligned}$$

Since $\mu_Y = 1 = 4a$, we must have $a = 1/4$.

Problem 10.2.4

(a) Note that $|H(f)| = 1$. This implies $S_{\hat{M}}(f) = S_M(f)$. Thus the average power of $\hat{M}(t)$ is

$$\hat{q} = \int_{-\infty}^{\infty} S_{\hat{M}}(f)\, df = \int_{-\infty}^{\infty} S_M(f)\, df = q$$

(b) The average power of the upper sideband signal is

$$\begin{aligned}
E\left[U^2(t)\right] = E&\left[M^2(t)\cos^2(2\pi f_c t + \Theta)\right] \\
&- E\left[2M(t)\hat{M}(t)\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t + \Theta)\right] \\
&+ E\left[\hat{M}^2(t)\sin^2(2\pi f_c t + \Theta)\right]
\end{aligned}$$

To find the expected value of the random phase cosine, for an integer $n \neq 0$, we evaluate

$$\begin{aligned}
E[\cos(2\pi f_c t + n\Theta)] &= \int_{-\infty}^{\infty} \cos(2\pi f_c t + n\theta)\, f_\Theta(\theta)\, d\theta \\
&= \int_0^{2\pi} \cos(2\pi f_c t + n\theta)\frac{1}{2\pi} d\theta \\
&= \frac{1}{2n\pi}\sin(2\pi f_c t + n\theta)\big|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\sin(2\pi f_c t + 2n\pi) - \sin(2\pi f_c t)\right) = 0
\end{aligned}$$

Similar steps will show that for any integer $n \neq 0$, the random phase sine also has expected value

$$E[\sin(2\pi f_c t + n\Theta)] = 0$$

Using the trigonometric identity $\cos^2\phi = (1+\cos 2\phi)/2$, we can show

$$E\left[\cos^2(2\pi f_c t+\Theta)\right] = E\left[\frac{1}{2}\left(1+\cos(2\pi(2f_c)t+2\Theta)\right)\right] = 1/2$$

Similarly,

$$E\left[\sin^2(2\pi f_c t+\Theta)\right] = E\left[\frac{1}{2}\left(1-\cos(2\pi(2f_c)t+2\Theta)\right)\right] = 1/2$$

In addition, the identity $2\sin\phi\cos\phi = \sin 2\phi$ implies

$$E[2\sin(2\pi f_c t+\Theta)\cos(2\pi f_c t+\Theta)] = E[\cos(4\pi f_c t+2\Theta)] = 0$$

Since $M(t)$ and $\hat{M}(t)$ are independent of Θ, the average power of the upper sideband signal is

$$\begin{aligned}
E\left[U^2(t)\right] &= E\left[M^2(t)\right]E\left[\cos^2(2\pi f_c t+\Theta)\right] + E\left[\hat{M}^2(t)\right]E\left[\sin^2(2\pi f_c t+\Theta)\right] \\
&\quad - E\left[M(t)\hat{M}(t)\right]E[2\cos(2\pi f_c t+\Theta)\sin(2\pi f_c t+\Theta)] \\
&= q/2 + q/2 + 0 = q
\end{aligned}$$

Problem 10.3.3
Theorem 10.9 which states

$$S_{XY}(f) = H(f)S_X(f)$$

(a) From Table 10.1, we observe that

$$S_X(f) = \frac{8}{16+(2\pi f)^2} \qquad H(f) = \frac{1}{7+j2\pi f}$$

(b) From Theorem 10.9,

$$S_{XY}(f) = H(f)S_X(f) = \frac{8}{[7+j2\pi f][16+(2\pi f)^2]}$$

(c) To find the cross correlation, we need to find the inverse Fourier transform of $S_{XY}(f)$. A straightforward way to do this is to use a partial fraction expansion of $S_{XY}(f)$. That is, by defining $s = j2\pi f$, we observe that

$$\frac{8}{(7+s)(4+s)(4-s)} = \frac{-8/33}{7+s} + \frac{1/3}{4+s} + \frac{1/11}{4-s}$$

Hence, we can write the cross spectral density as

$$S_{XY}(f) = \frac{-8/33}{7+j2\pi f} + \frac{1/3}{4+j2\pi f} + \frac{1/11}{4-j\pi f}$$

Unfortunately, terms like $1/(a - j2\pi f)$ do not have an inverse transforms. The solution is to write $S_{XY}(f)$ in the following way:

$$\begin{aligned} S_{XY}(f) &= \frac{-8/33}{7+j2\pi f} + \frac{8/33}{4+j2\pi f} + \frac{1/11}{4+j2\pi f} + \frac{1/11}{4-j2\pi f} \\ &= \frac{-8/33}{7+j2\pi f} + \frac{8/33}{4+j2\pi f} + \frac{8/11}{16+(2\pi f)^2} \end{aligned}$$

Now, we see from Table 10.1 that the inverse transform is

$$R_{XY}(\tau) = -\frac{8}{33}e^{-7\tau}u(\tau) + \frac{8}{33}e^{-4\tau}u(\tau) + \frac{1}{11}e^{-4|\tau|}$$

Problem 10.4.1

(a) Since $C_X(t_1, t_2 - t_1) = \rho\sigma_1\sigma_2$, the covariance matrix is

$$\mathbf{C} = \begin{bmatrix} C_X(t_1,0) & C_X(t_1,t_2-t_1) \\ C_X(t_2,t_1-t_2) & C_X(t_2,0) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Since $\mathbf{C}$ is a 2×2 matrix, it has determinant $|\mathbf{C}| = \sigma_1^2\sigma_2^2(1-\rho^2)$.

(b) Is is easy to verify that

$$\mathbf{C}^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_1^2} \end{bmatrix}$$

(c) The general form of the multivariate density for $X(t_1), X(t_2)$ is

$$f_{X(t_1),X(t_2)}(x_1,x_2) = \frac{1}{(2\pi)^{k/2}|\mathbf{C}|^{1/2}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu}_X)^\top \mathbf{C}^{-1}(\bar{x}-\bar{\mu}_X)}$$

where $k=2$ and $\bar{x} = [x_1\ x_2]^\top$ and $\bar{\mu}_X = [\mu_1\ \mu_2]^\top$. Hence,

$$\frac{1}{(2\pi)^{k/2}|\mathbf{C}|^{1/2}} = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}}$$

Furthermore, the exponent is

$$\begin{aligned} -\frac{1}{2}(\bar{x}-\bar{\mu}_X)^\top \mathbf{C}^{-1}(\bar{x}-\bar{\mu}_X) &= -\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_1^2} \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \\ &= -\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \end{aligned}$$

Plugging in each piece into the joint PDF $f_{X(t_1),X(t_2)}(x_1,x_2)$ given above, we obtain the bivariate Gaussian PDF.

Problem 10.5.3

(a) Since $S_W(f) = 10^{-15}$ for all f, $R_W(\tau) = 10^{-15}\delta(\tau)$.

(b) Since Θ is independent of $W(t)$,

$$E[V(t)] = E[W(t)\cos(2\pi f_c t + \Theta)] = E[W(t)]E[\cos(2\pi f_c t + \Theta)] = 0$$

(c) We cannot initially assume $V(t)$ is WSS so we first find

$$\begin{aligned}
R_V(t,\tau) &= E[V(t)V(t+\tau)] \\
&= E[W(t)\cos(2\pi f_c t + \Theta)W(t+\tau)\cos(2\pi f_c (t+\tau) + \Theta)] \\
&= E[W(t)W(t+\tau)]E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta)] \\
&= 10^{-15}\delta(\tau)E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta)]
\end{aligned}$$

We see that for all $\tau \neq 0$, $R_V(t, t+\tau) = 0$. Thus we need to find the expected value of

$$E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta)]$$

only at $\tau = 0$. However, its good practice to solve for arbitrary τ:

$$\begin{aligned}
&E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta)] \\
&= \frac{1}{2}E[\cos(2\pi f_c \tau) + \cos(2\pi f_c (2t+\tau) + 2\Theta)] \\
&= \frac{1}{2}\cos(2\pi f_c \tau) + \frac{1}{2}\int_0^{2\pi} \cos(2\pi f_c (2t+\tau) + 2\theta)\frac{1}{2\pi}\,d\theta \\
&= \frac{1}{2}\cos(2\pi f_c \tau) + \frac{1}{2}\sin(2\pi f_c (2t+\tau) + 2\theta)\Big|_0^{2\pi} \\
&= \frac{1}{2}\cos(2\pi f_c \tau) + \frac{1}{2}\sin(2\pi f_c (2t+\tau) + 4\pi) - \frac{1}{2}\sin(2\pi f_c (2t+\tau)) \\
&= \frac{1}{2}\cos(2\pi f_c \tau)
\end{aligned}$$

Consequently,

$$R_V(t,\tau) = \frac{1}{2}10^{-15}\delta(\tau)\cos(2\pi f_c \tau) = \frac{1}{2}10^{-15}\delta(\tau)$$

(d) Since $E[V(t)] = 0$ and since $R_V(t,\tau) = R_V(\tau)$, we see that $V(t)$ is a wide sense stationary process. Since $L(f)$ is a linear time invariant filter, the filter output $Y(t)$ is also a wide sense stationary process.

(e) The filter input $V(t)$ has power spectral density $S_V(f) = \frac{1}{2}10^{-15}$. The filter output has power spectral density

$$S_Y(f) = |L(f)|^2 S_V(f) = \begin{cases} 10^{-15}/2 & |f| \leq B \\ 0 & \text{otherwise} \end{cases}$$

The average power of $Y(t)$ is

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-B}^{B} \frac{1}{2}10^{-15}\,df = 10^{-15}B$$

Problem 10.5.4
and $Y(t)$ are the input andoutput of a linear time invariant filter $h(u)$. In that case,

$$Y(t) = \int_0^t N(u)\,du = \int_{-\infty}^{\infty} h(t-u)N(u)\,du$$

For the above two integrals to be the same, we must have

$$h(t-u) = \begin{cases} 1 & 0 \le t-u \le t \\ 0 & \text{otherwise} \end{cases}$$

Making the substitution $v = t-u$, we have

$$h(v) = \begin{cases} 1 & 0 \le v \le t \\ 0 & \text{otherwise} \end{cases}$$

Thus the impulse response $h(v)$ depends on t. That is, the filter response is linear but not time invariant. Since Theorem 10.7 requires that $h(t)$ be time invariant, this example does not violate the theorem.

Problem 10.5.5
process since it is the output of a linear filter with Gaussian input process $N(t)$. We observe that $E[Y(t)] = \int_0^t E[N(u)]\,du = 0$. The autocorrelation funciton of the output is

$$\begin{aligned} R_Y(t,\tau) &= E[Y(t)Y(t+\tau)] \\ &= E\left[\left(\int_0^t N(u)\,du\right)\left(\int_0^{t+\tau} N(v)\,dv\right)\right] \\ &= \int_0^t \int_0^{t+\tau} E[N(u)N(v)]\,dv\,du \end{aligned}$$

Since $N(t)$ is a White noise process,

$$E[N(u)N(v)] = R_N(u, v-u) = \alpha\delta(v-u)$$

This implies

$$R_Y(t,\tau) = \alpha \int_0^t \left(\int_0^{t+\tau} \delta(v-u)\,dv\right) du$$

If $\tau \ge 0$, then for each $u \in [0,t]$ there is $v \in [0, t+\tau]$ such that $v = u$ and $\int_0^{t+\tau} \delta(v-u)\,dv = 1$. This implies

$$R_Y(t,\tau) = \alpha \int_0^t du = \alpha t$$

If $\tau < 0$, then we write

$$R_Y(t,\tau) = \alpha \int_0^{t+\tau} \left(\int_0^t \delta(v-u)\,du\right) dv$$

For each $v \in [0, t+\tau]$ there is $u \in [0, t]$ such that $u = v$ and $\int_0^t \delta(v-u)\,du = 1$. This implies

$$R_Y(t,\tau) = \alpha \int_0^{t+\tau} dv = \alpha(t+\tau)$$

The complete expression for the autocorrelation of $Y(t)$ is

$$R_Y(t,\tau) = \alpha \min(t, t+\tau)$$

In Chapter 6, we found that a Brownian motion process $X(t)$ is a zero mean Gaussian process. In addition, in Example 6.17, we found that a Brownian motion process $X(t)$ has autocorrelation function

$$R_X(t,\tau) = \alpha \min(t, t+\tau)$$

Since a Gaussian process is completely specified by the mean $E[X(t)]$ and the autocorrelation $R_X(t,\tau)$, we can conclude that $Y(t)$ must be a Gaussian process.

Another way to interpret this result is to write for $t > s$, the increment is

$$Y(t) - Y(s) = \int_0^t N(u)\,du - \int_0^s N(u)\,du = \int_s^t N(u)\,du$$

For each $v \in [s,t]$, $N(v)$ is independent of $N(u)$ for any $u \in [0,s]$. Thus for any $s' \leq s$, $Y(s') = \int_0^{s'} N(u)\,du$ is independent of $Y(t) - Y(s)$. Hence $Y(t)$ is a zero mean Gaussian process with independent increments and $Y(0) = 0$, which is Definition 6.11 for the Brownian motion process.

Problem 10.6.4

We start with Theorem 10.13:

$$\begin{aligned} R_Y[n] &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j R_X[n+i-j] \\ &= R_X[n-1] + 2R_X[n] + R_X[n+1] \end{aligned}$$

First we observe that for $n \leq -2$ or $n \geq 2$,

$$R_Y[n] = R_X[n-1] + 2R_X[n] + R_X[n+1] = 0$$

This suggests that $R_X[n] = 0$ for $|n| > 1$. In addition, we have the following facts:

$$\begin{aligned} R_Y[0] &= R_X[-1] + 2R_X[0] + R_X[1] = 2 \\ R_Y[-1] &= R_X[-2] + 2R_X[-1] + R_X[0] = 1 \\ R_Y[1] &= R_X[0] + 2R_X[1] + R_X[2] = 1 \end{aligned}$$

A simple solution to this set of equations is $R_X[0] = 1$ and $R_X[n] = 0$ for $n \neq 0$.