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332:501 | Systems Analysis | Fall 2002

Extra Homework I

1. **Power Control**

   You are given the following power control iteration where $\mathbf{p}(n)$ is a power vector at step n, G is a target signal to interference ratio and b is a background noise level. Assume both $G, b > 0$ and $\mathbf{p}(n)$ as well.

   $$\mathbf{p}(n+1) = Ga \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{p}(n) + Gb \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{1}$$

   For what values of G, a and b is this mapping a contraction?

   SOLN: Let $T[]$ be the mapping so we evaluate (using the euclidean norm for convenience)

   $$|T[\mathbf{p}_1] - T[\mathbf{p}_2]| = Ga \left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (\mathbf{p}_1 - \mathbf{p}_2) \right| \leq Ga \left\| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\| |\mathbf{p}_1 - \mathbf{p}_2| = Ga |\mathbf{p}_1 - \mathbf{p}_2| \tag{2}$$

   So, as long as $Ga < 1$ we have a contraction (under the euclidean norm).

   And though I did not ask what the fixed point was,

   $$\begin{bmatrix} -1 & Ga \\ Ga & -1 \end{bmatrix} \mathbf{p}^* = -Gb \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3}$$

   or

   $$\mathbf{p}^* = \frac{Gb(1+Ga)}{1 - G^2a^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tag{4}$$

   Now of course, this is a relatively boring situation. The real fun is to show that the procedure is always convergent when we bound $|\mathbf{p}|_\infty \leq P$ (power constraint). It turns out that though it's not a contraction, the iteration always converges regardless of the values of a and G (and b).

2. **Maps and Bounding**

   Given

   $$\rho_1(A(x), A(y)) \leq K_1 \rho_1(x, y) \tag{5}$$

   is

   $$\rho_2(A(x), A(y)) \overset{?}{\leq} K_2 \rho_2(x, y) \tag{6}$$

   for some suitable K_2?

   SOLN: Another few minutes on the drive home yielded the following counterexample which shows that if a mapping is bounded in one metric space it's not necessarily bounded in another.

Let $\rho_1(x,y) = |x-y|$ for $x, y \in [-1,1]$. Then for $0 < a < 1$ let

$$A(x) = \begin{cases} x/a & |x| < a \\ 1 & \text{otherwise} \end{cases}$$

$A()$ is easily shown to be bounded for this metric space with bounding constant $1/a$. Now define

$$\rho_2(x,y) = \left| \frac{x}{1-|x|} - \frac{y}{1-|y|} \right|$$

Obviously it's non-negative and zero for $x = y$. Equally obvious it's symmetric ($\rho_2(x,y) = \rho_2(y,x)$). And we also have

$$\rho_2(x,y) = \left| \frac{x}{1-|x|} - \frac{z}{1-|z|} + \frac{z}{1-|z|} - \frac{y}{1-|y|} \right| \leq \rho_2(x,z) + \rho_2(z,y)$$

so $\rho_2()$ is a distance measure. However, suppose $x = -a$ and $y = a$. $\rho_2(x,y)$ is finite while $\rho_2(A(x), A(y))$ is infinite. So **there is no bounding constant such that** $\rho_2(A(x), A(y)) \leq K_2 \rho_2(x,y)\ \forall x, y \in [-1,1]$.