The Convolution and Delta Functions

We are starting with the following expressions:

$$T[\delta(t-t_0)] = h(t, t_0),$$

$$x(t-t_0) = \int_{-\infty}^{+\infty} x(\tau)\cdot\delta(t-t_0-\tau)\,d\tau,$$

$$T[a_1x_1(t)+a_2x_2(t)] = a_1T[x_1(t)]+a_2T[x_2(t)].$$

So, we are assuming that the system is *linear*.

Now, we will apply operator T to x(t):

$$T[x(t-t_0)]\text{¿} = T\left[\int_{-\infty}^{+\infty} x(\tau)\cdot\delta(t-t_0-\tau)\,d\tau\right]$$

Since an integral is linear operator (as well as T is), we can change their order:

$$T[x(t-t_0)] = \int_{-\infty}^{+\infty} T[x(\tau)\cdot\delta(t-t_0-\tau)]\,d\tau$$

The next one is the critical part. Remember that any function multiplied with $\delta(t\text{-}t_0\text{-}\tau)$ equals zero in all points except for $t\text{-}t_0 = \tau$. Because of that I can treat $x(\tau)$ **as a constant** and put it in front of T. Maybe I can express this in other way: We are applying operator T on t as a variable; since $x(\tau)$ does not depend on t we can skip it.

$$\int_{-\infty}^{+\infty} x(\tau)\cdot T[\delta(t-t_0-\tau)]\,d\tau = \int_{-\infty}^{+\infty} x(\tau)\cdot h(t-\tau, t_0)\,d\tau$$

The last step: Assume that the system is *time invariant*, meaning:

$$y(t) = T[x(t)] \quad\Rightarrow\quad y(t-t_0) = T[x(t-t_0$$

$$h(t, t_0) = h(t-t_0) = T[\delta(t-t_0)],$$

$$T[x(t)] = \int_{-\infty}^{+\infty} x(\tau)\cdot h(t-\tau)\,d\tau = f(t) * h$$