

You have the class period to answer the following three (3) questions. The point values are as shown and the examination is open book. The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT.

GOOD LUCK!

1. (30 points) **Contraction Maps and Matrices**

This problem IS ALMOST identical to one which appeared on the 1998 first quiz. Why so easy? Because I want you to CALM DOWN and work through something you know for the first problem. All vectors and matrices are assumed to be elements of \mathfrak{R}^N and $\mathfrak{R}^N \times \mathfrak{R}^N$ respectively.

(a) (15 points) We can state with certainty that the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is convergent if:

- i. $\|\mathbf{A}\|_1 < 1$
- ii. $\|\mathbf{A}\|_2 < 1$
- iii. $\|\mathbf{A}\|_\infty < 1$
- iv. (a) or (b) or (c) [logical or]
- v. None of the above

(no partial credit on this part).

The sequence is convergent if it's a contraction under any metric. So iv is the right answer.

(b) (15 points) Suppose for some $\mathbf{x}^* \neq \mathbf{0}$ we have $\mathbf{x}^* = \mathbf{A}\mathbf{x}^*$. Can the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ be a contraction over \mathfrak{R}^N ? Why/why not?

HINT: Remember that \mathbf{A} is a matrix.

If $\mathbf{A}\mathbf{x} = \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$ then there are at least two fixed points since $\mathbf{A}\mathbf{0} = \mathbf{0}$. Can't be a contraction because the fixed point is not unique.

2. (30 points) **Some Indirect Linear Algebra:**

You are given a matrix:

$$\mathbf{A} = \begin{bmatrix} -2 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & -2 & \frac{1}{2} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & \cdots & \frac{1}{2} & -2 \end{bmatrix}$$

What can (or can't) you say about the quantity $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ for ANY suitably sized real vector \mathbf{x} ? Since \mathbf{A} is a symmetric matrix, its eigenvalues are real.

You use Gershgorin to show that the eigenvalues of \mathbf{A} must reside in a closed ball of radius 1 about $(-2, 0)$ in the complex plane. Thus all eigenvalues have negative real parts. The matrix is symmetric and real so its eigenvalues are real and negative. The matrix is negative definite which means that $\mathbf{x}^\top \mathbf{A} \mathbf{x} < 0$.

3. (40 points) Multivariable Optimization:

You are given functions

$$\mathcal{L}(\mathbf{x}) = \sum_{n=1}^N x_n \int_{x_{n-1}}^{x_n} g(s) ds$$

and

$$\mathcal{D}(\mathbf{x}) = \sum_{n=1}^N n \int_{x_{n-1}}^{x_n} g(s) ds$$

where $g(s)$ is a continuous non-increasing non-negative, unit area function on $[0, X]$ where $X > 0$. You can think of $g(s)$ as a probability density function if you'd like. The x_n are non-negative and increasing so that $x_n \geq x_{n-1}$ and we require $x_N = X$. Please show that the function

$$\mathcal{G} = \mathcal{L} + \alpha \mathcal{D}$$

is convex in \mathbf{x} when α is chosen such that

$$(x_{n+1} - x_n + \alpha)g(x_n) = \int_{x_{n-1}}^{x_n} g(s) ds \quad (1)$$

HINT: You'll REALLY want to use the λ -slice method we talked about in class.

The second partials of \mathcal{G} are

$$\frac{\partial^2 \mathcal{G}}{\partial x_i \partial x_j} = \begin{cases} (x_i - x_{i+1} - \alpha) \frac{dg(x_i)}{dx_i} + 2g(x_i) & j = i \\ -g(x_i) & j = i + 1 \\ -g(x_{i-1}) & j = i - 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Given \mathbf{x} and \mathbf{y} , let $\mathbf{z}(\lambda) = \lambda \mathbf{x} + (1 - \lambda) \mathbf{y}$. We will show that $\mathcal{G}(\mathbf{z}(\lambda))$ is convex in λ over $0 \leq \lambda \leq 1$ for all admissible \mathbf{x}, \mathbf{y} . Note that it is easily shown that if \mathbf{x} and \mathbf{y} are admissible (i.e., $x_n \leq x_{n+1}$ and $y_n \leq y_{n+1}$) then \mathbf{z} is admissible as well.

Let $\Delta = \mathbf{x} - \mathbf{y}$ so that $\mathbf{z} = \lambda \Delta + \mathbf{y}$. We then have

$$\frac{\partial^2 \mathcal{G}(\mathbf{z})}{\partial \lambda^2} = \sum_{i,j=1}^N \frac{\partial^2 \mathcal{G}(\mathbf{z})}{\partial x_i \partial x_j} \Delta_i \Delta_j \quad (3)$$

Using equation (2) we obtain,

$$\begin{aligned}
\frac{\partial^2 \mathcal{G}}{\partial \lambda^2} &= \sum_{i=1}^N \frac{\partial^2 \mathcal{G}(\mathbf{z})}{\partial x_i^2} \Delta_i^2 + 2 \sum_{i=1}^{N-1} \frac{\partial^2 \mathcal{G}(\mathbf{z})}{\partial x_i \partial x_{i+1}} \Delta_i \Delta_{i+1} \\
&= \sum_{i=1}^N (z_i - z_{i+1} - \alpha) g'(z_i) \Delta_i^2 + 2 \sum_{i=1}^N g(z_i) \Delta_i^2 - 2 \sum_{i=1}^{N-1} g(z_i) \Delta_i \Delta_{i+1} \\
&= \sum_{i=1}^N (z_i - z_{i+1} - \alpha) g'(z_i) \Delta_i^2 + g(z_1) \Delta_1^2 + g(z_N) \Delta_N^2 \\
&\quad + \sum_{i=1}^{N-1} g(z_i) (\Delta_i - \Delta_{i+1})^2
\end{aligned} \tag{4}$$

For $\alpha \geq 0$ we have $z_n - z_{n+1} - \alpha \leq 0$ which implies that $\frac{\partial^2 \mathcal{G}}{\partial \lambda^2} \geq 0$ and \mathcal{G} is convex when $\alpha \geq 0$. The same holds true for α chosen to satisfy equation (1) owing to the positivity of $g(\cdot)$.