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330:501 | Systems Analysis | Fall 1997

FINAL EXAMINATION
December 16, 1997

You have THREE WHOLE HOURS to answer the questions given. The point values are as shown. You are allowed six sides of an 8.5 X 11 sheets of notes for reference. The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT.

GOOD LUCK!

1. *(120 points)* **Rutgera Univera and the Charging Capacitor:**

   Rutgera Univera, the world famous Rutgers graduate student, is an energy conscious scientist and has found that many electronic appliances are filled with capacitors which must be charged. Inefficient charging of capacitors leads to energy loss and this is anathema to Rutgera. She has therefore decided to design "green" drive circuits which will efficiently charge capacitors. Your job is to help her.

   (a) *(10 points)* Please derive a differential equation which relates $u(t)$ and $v(t)$ for the circuit of FIGURE 1a. DO NOT SOLVE the equation. Please be careful. If you get this simple part wrong you will get the rest of the problem wrong and woe to you!

   $$v + RC\dot{v} = u$$

   (b) *(10 points)* Please derive an expression for the current driven into the circuit $i(t)$.

   $$i = C\dot{v}$$

   (c) *(10 points)* Assume that the capacitor has no charge at $t = 0$ and we desire $v(t_1) = V$. Now define the energy applied by the input source $u(t)$ as

   $$E = \int_0^{t_1} u(t)i(t)dt$$

   where $i(t)$ is the current flowing out of the source. Rewrite E in terms of $v(t)$ and $\dot{v}(t)$. That is, derive an expression for E as

   $$E = \int_0^{t_1} F[t, v(t), \dot{v}(t)]dt$$

   where $F[]$ is some function.
   HINT: $F[]$ has no *explicit* dependence on t since the system is time invariant.

   $$E = \int_0^{t_1} (v(t) + RC\dot{v}(t))C\dot{v}(t)dt$$

(d) *(10 points)* Find the trajectory $v(t)$ which minimizes E with $v(0) = 0$ and $v(t_1) = V$ by carefully solving the differential equation:

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{v}} - \frac{\partial F}{\partial v} = 0$$

Notice the initial TOTAL derivative (not partial) and be sure to use the chain rule to expand.

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{v}} = \frac{d}{dt}[Cv(t) + 2RC^2\dot{v}(t)] = C\dot{v}(t) + 2RC^2\ddot{v}(t)$$

$$\frac{\partial F}{\partial v} = C\dot{v}(t)$$

so we have

$$2RC^2\ddot{v}(t) = 0$$

which means that $v(t) = \alpha t + \beta$. Matching up initial and final conditions yields

$$v(t) = \frac{V}{t_1}t$$

(e) *(10 points)* Find the $u(t)$ which produces the desired trajectory $v(t)$. Sketch your $u(t)$ and corresponding $v(t)$.

$$u(t) = \frac{V}{t_1}(t + RC)$$

We'll forego the sketch here. Pretty clear it's linear with an initial offset for $u(t)$ and linear for $v(t)$.

(f) *(10 points)* Find E in the limit as $t_1 \to \infty$, compare to the E obtained when $u(t)$ is step of height V and comment.

For the optimal case $u(t) = \frac{V}{t_1}(t + RC)$ and $i(t) = C\frac{V}{t_1}$ so that

$$E = \int_0^{t_1} C\left(\frac{V}{t_1}\right)^2 (t + RC)dt = \frac{1}{2}C\left(\frac{V}{t_1}\right)^2 [(t_1 + RC)^2 - R^2C^2]$$

which is $E = \frac{1}{2}CV^2$ as $t_1 \to \infty$. This is very satisfying since this is exactly the amount of energy stored in the capacitor – you can't use any less energy than that to charge it. When using a step input, the capacitor voltage is $v(t) = V(1 - e^{-t/RC})$ so the current is $\frac{V}{R}e^{-t/RC}$. The energy is then

$$E = \int_0^{\infty} \frac{V^2}{R}(e^{-2t/RC})dt = CV^2$$

which is twice that of the optimal method.

It's also interesting to note that as t_1 is made very small you have to expend a lot of energy (E goes as $1/t_1^2$) to charge the capacitor quickly. This result has obvious implications for high speed circuits and is known to anyone who has physically touched an old Intel Pentium processor while it was running.

(g) *(10 points)* Now consider the circuit of FIGURE 1b with two capacitors. Please derive a state-space description of the system using v_1 and v_2 as your state variables. Once again, DON'T BLOW THIS PART OR YOU'RE IN BIG TROUBLE.

KVL/KCL: $u = RC\dot{v}_1 + RC\dot{v}_2 + v_1$ and $v_1 = RC\dot{v}_2 + v_2$.

$$\dot{v} = \begin{bmatrix} -\frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} v + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} u$$

(h) *(10 points)* Is this system controllable? Why?/Why not?

$$K = \begin{bmatrix} \frac{1}{RC} & -\frac{2}{RC} \\ 0 & \frac{1}{RC} \end{bmatrix}$$

which clearly has full rank.

(i) *(10 points)* Assume $v_1(0) = RCV/t_1$ and $v_2(0) = 0$. Logically apply your previous results to obtain an input $u(t)$ which optimally charges the capacitors to $v_1(t_1) = V(1+RC/t_1)$ and $v_2(t_1) = V$. What is the energy expended by your method as $t_1 \to \infty$?

What we'll try to do is have the capacitors charge linearly (constant current). Therefore we want $v_1(t) = RCV/t_1 + Vt/t_1$ and $v_2(t) = Vt/t_1$. Therefore we must have $i(t) = 2CV/t_1$. As a check we see that using KVL we have $v_1(t) = RCV/t_1 + Vt/t_1$ which gibes with our assumptions about v_1 and v_2. Using KVL again we have

$$u(t) = 3RCV/t_1 + Vt/t_1$$

The energy expended is

$$E = \int_0^{t_1} 2CV/t_1(3RCV/t_1 + Vt/t_1)dt = 6RC^2V^2/t_1CV^2$$

which is simply CV^2 in the limit, the mininum amount of energy possible to expend charging the capacitors.

2. *(60 points)* **Otilia's Problem:**

Consider the following system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

(a) *(20 points)* Is this system stable?

All eigenvalues have positive real parts so the system is UNSTABLE.

(b) *(20 points)* Is this system controllable?

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

which is full rank so the system though unstable is COMPLETELY CONTROLLABLE.

(c) *(20 points)* PROVE that this system CANNOT be driven along all desired state trajectory $x(t)$. State all assumptions.

Note that the system is a decoupled system since x_1 has no effect on x_2 and vice versa. Writing out the equations we have $\dot{x}_1 = x_1 + u$ and $\dot{x}_2 = 2x_2 + u$. Suppose we want to choose u so that we have $x_1 = x_2 = C$ where C is a constant. We end up with two conflicting values of u (C and $2C$ respectively) so NO CAN DO even though the system is completely controllable!

3. *(60 points)* **Lost in Space:**

Recall that the equations of motion for the satellite problem are

$$\frac{d^2\theta}{dt^2} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$

and

$$\frac{d^2r}{dt^2} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2$$

TURNS OUT THESE EQUATIONS HAD TYPOS. SHOULD HAVE BEEN

$$\frac{d^2r}{dt^2} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$

and

$$\frac{d^2\theta}{dt^2} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2$$

The solutions below are worked out for the first case. For a solution to the real physical system, please check the text.

(a) *(10 points)* Recast these two second-order equations as a set of four first order equations.
Since we have two derivatives in r and θ the natural thing to do is to define $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$ and $x_4 = \dot{\theta}$. We then have

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\tfrac{2x_4x_2}{x_1} + \tfrac{1}{x_1}u_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= x_1x_4^2 - \tfrac{k}{x_1^2} + u_1
\end{aligned}$$

(b) *(10 points)* Verify that the solution $r = R$, $\theta = \omega t$ is a solution to the system when $u_1 = u_2 = 0$ where R and ω are constants. Find the required relationship between R and ω.
The equations in $\dot{x}_1$ and $\dot{x}_3$ are obvious and always satisfied. The equation in $\dot{x}_2$ is satisfied since x_2 and $\dot{x}_2$ are zero when $r = R$. We are then left with $x_1x_4^2 = \frac{k}{x_1^2}$ which implies $R\omega^2 = \frac{k}{R^2}$ so that $\omega^2 = \frac{k}{R^3}$.

(c) *(10 points)* Linearize the system about this ORBIT TRAJECTORY and cast it in statespace form. Again be careful here since if you screw this part up you're probably screwed for the rest of this problem.
The equations in $\dot{x}_1$ and $\dot{x}_3$ are already linear. We then have

$$\dot{x}_2 = -\frac{2\omega}{R}x_2 + \frac{1}{R}u_2$$

and

$$\dot{x}_4 = x_1\omega^2 + 2\omega R x_4 + \frac{2k}{R^3}x_1 + u_1$$

To make life easier we'll put this in statespace form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 - \omega \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{2\omega}{R} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2k}{R^3} + \omega^2 & 0 & 0 & 2\omega R \end{bmatrix} \begin{bmatrix} x_1 - R \\ x_2 \\ x_3 - \omega t \\ x_4 - \omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R} \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

We can also write this in terms of incremental variables to obtain

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{2\omega}{R} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2k}{R^3} + \omega^2 & 0 & 0 & 2\omega R \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R} \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(d) *(10 points)* Determine whether the linearized system is stable about this ORBIT TRAJECTORY.

Need to find the eigenvalues. First form

$$\begin{bmatrix} -s & 1 & 0 & 0 \\ 0 & -s+\frac{2\omega}{R} & 0 & 0 \\ 0 & 0 & -s & 1 \\ \frac{2k}{R^3}+\omega^2 & 0 & 0 & -s+2\omega R \end{bmatrix}$$

The determinant is

$$s^2(\frac{2\omega}{R}-s)(2\omega R-s)$$

There are two positive roots so the system is unstable.

(e) *(10 points)* Is the linearized system controllable using inputs u_1 and u_2?

A few ways to go here. The tried and true is to compute K. But we'll only need to look at the first six columns because that's enough to see that the K matrix has full rank.

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2\omega}{R^2} & 0 & \frac{4\omega^2}{R^3} \\ 0 & \frac{1}{R} & 0 & \frac{2\omega}{R^2} & 0 & \frac{4\omega^2}{R^3} & 0 & \frac{8\omega^3}{R^4} \\ 0 & 0 & 1 & 0 & 2\omega R & 0 & 4\omega^2R^2 & 0 \\ 1 & 0 & 2\omega R & 0 & 4\omega^2R^2 & 0 & 8\omega^3R^3 & (\frac{2k}{R^3}+\omega^2)\frac{2\omega}{R^2} \end{bmatrix}$$

(f) *(10 points)* Suppose we are only allowed to observe the radial position r. Is the linearized system observable?

We now have an implied output equation of

$$y = [\; 1 \quad 0 \quad 0 \quad 0 \;]z$$

Form the observability matrix M

$$M = [C^T|A^TC^T|(A^T)^2C^T|(A^T)^3C^T] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2\omega}{R} & \frac{4\omega^2}{R^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which can never have full rank since two rows are zero.

4. *(60 points)* **Sherlock Holmes Strikes Again:**

Consider the system depicted in FIGURE 2 with velocity INPUT v and OBSERVED force F. Is this system observable?

You have to first derive the equations of motion for the system. Let v_m be the mass velocity. The difference in velocity across the springs is $v - v_m$ for the righthand spring and $v_m - v$ for the lefthand spring. Writing force balance yields

$$m\ddot{v}_m = 2k(v - v_m)$$

The force applied to the box by the springs is our output force. The derivative of the force applied by the springs is $\dot{F} = -2k(v - v_m)$

Using v_m and $\dot{v}_m$ as the state variables, the implied system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{2k}{m} \end{bmatrix} v$$

$$\dot{F} = \begin{bmatrix} 2k & 0 \end{bmatrix} x - 2kv$$

Define a new output variable as $Q = \dot{F} + 2kv$ and we have

$$Q = \begin{bmatrix} 2k & 0 \end{bmatrix} x$$

Clearly if we know $F(t)$ and $v(t)$ we can determine Q. So if the system with Q as output is observable, then our original system is observable as well.

The observability matrix:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{2k}{m} \end{bmatrix}$$

which has full rank so the system is observable.

5. *(60 points)* **Old Balls Don't Die, They Just Roll More Slowly:**

   Consider the system depicted in FIGURE 3. A mass M moves friction-free on a track between two identical elastic/damping bumpers as shown. This is somewhat analogous to the marble rolling along the flat-bottomed bowl problem considered in class, only it's simpler to analyze. You can thank Wolfgang for pointing this out.

   (a) *(20 points)* Define state variables for the system and then derive the differential equations of motion for this nonlinear system.

   The state variables are position and velocity. Reference x to the midpoint between the bumpers and we have three regions:

   - Region I is $x < -L/2$
   - Region II is $-L/2 < x < L/2$
   - Region III is $x > L/2$

   The equations of motion for region II are easy: $\ddot{x} = 0$ which implies constant velocity. For region I we have $m\ddot{x} = -K(x + L/2) - B\dot{x}$. Likewise for region III we have $m\ddot{x} = -K(x - L/2) - B\dot{x}$

   (b) *(20 points)* Precisely state the conditions under which a function $f(\mathbf{x})$ is a Lyapunov function for a system.

   For $f()$ to be Lyapunov it has to be a (i) continuous scalar function with a (ii) unique minimum which occurs at the fixed point. In addition, (iii) $f(x)$ must be nonincreasing along any trajectory x. That is $df/dt \leq 0$ for (marginal) stability and < 0 for asymptotic stability.

   (c) *(20 points)* Show that the energy in the system is a Lyapunov function IFF $L = 0$. Comment on the stability of this system for various values of L.

   This is a giveaway. The minimum energy is zero. The energy is zero $\forall x \in (-L/2, L/2)$ and $\dot{x} = 0$. Unless $L = 0$ there is no unique point in the statespace where the energy achieves it's minimum – so it's can't be a Lyapunov function unless $L = 0$.

   Now as for stability. The system is unstable unless $L = 0$. Why? First consider that there are an infinite number of fixed point. Drop the ball with zero velocity at any point on $(-L/2, L/2)$ and it stays there. However, for any of these points there is no nonzero neighborhood in statespace such that if you start close enough to the fixed point you never stray beyond some small distance from the fixed point. Here, if you have ANY velocity the ball eventually escapes the neighborhood of the fixed point.

   When $L = 0$ the system is asymptotically stable.

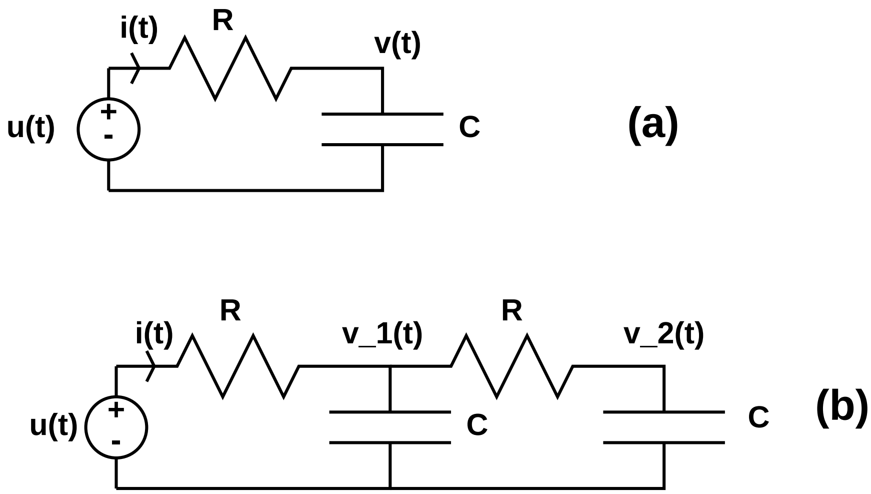

Figure 1: RC circuits figure for problem 1

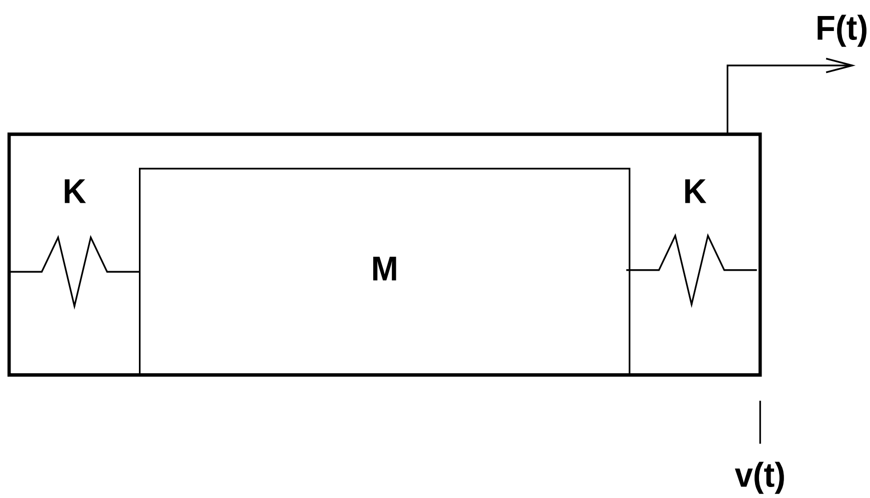

Figure 2: Masses and springs in a box

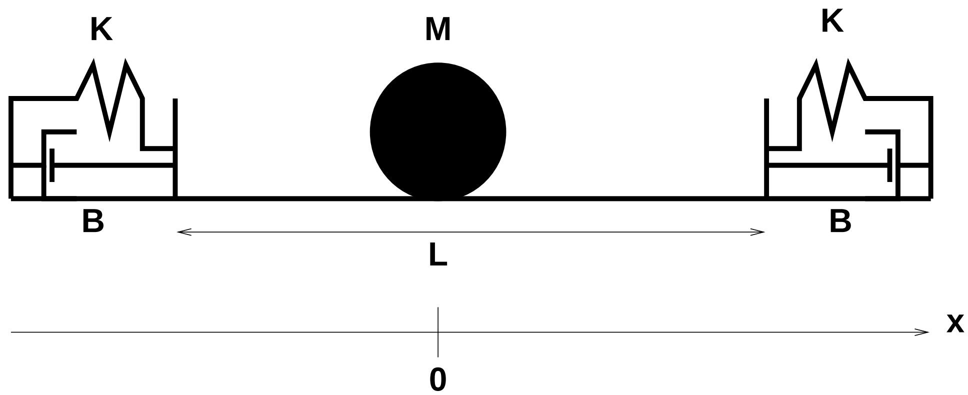

Figure 3: Ball and Bumpers