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College of Engineering
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330:501 Systems Analysis Fall 1997

SECOND EXAMINATION STATEMENT & SOLUTIONS

1. *(60 points)* **Rutgera Univera Strikes Again**

Rutgera Univera, the world famous Rutgers graduate student has been hired to stabilize a pendulum as shown in FIGURE 1.

This pendulum controls the detonator to an ancient nuclear device and must be carefully kept in the vertical upright position to keep the device from exploding. Needless to say, she's a little nervous about this job!

(a) *(10 points)* Show that Rutgera's equation of motion for the pendulum

$$g \sin\theta(t) = L\frac{d^2\theta}{dt^2} + \frac{d^2u}{dt^2}\cos\theta(t)$$

is correct where M is the mass of the pendulum bob, L is the length of the pendulum rod, $\theta(t)$ is the angle the pendulum rod makes with vertical and $u(t)$ is the displacement applied to the pivot point. g is the gravitational constant.

You may use direct methods or Lagrangians. In any event be careful.

Soln: There are at least two ways to use the Lagrangian. The potential energy is always $MgL\cos\theta$. The kinetic energy can be written in two ways. First, we could just use the rotational energy about the pivot, $\frac{1}{2}ML\dot{\theta}^2$. Then we have to consider the torque generated by acceleration in u of $-ML\frac{d^2u}{dt^2}\cos\theta$. Forming

$$L = \frac{1}{2}ML\dot{\theta}^2 - MgL\cos\theta$$

taking the appropriate derivatives and equating to the external torque yields the desired result.

Another way is to formulate the kinetic energy referenced to a fixed frame as $\frac{M}{2}[(\dot{u} + \dot{\theta}\cos\theta)^2 + (\dot{\theta}\sin\theta)^2]$. Now there is NO external torque to be reckoned with. So letting

$$L = \frac{M}{2}[(\dot{u} + \dot{\theta}L\cos\theta)^2 + (\dot{\theta}L\sin\theta)^2] - MgL\cos\theta$$

taking the derivatives and setting the result to zero gives us what we want.

A more direct approach (and therefore more comfortable for some) is to use an inertial frame and use $F = ma$ where F is the radial force applied by the pendulum rod.

First we have $x(t) = L\sin\theta(t) + u(t)$ and $y(t) = L\cos\theta(t)$ for the position of the mass. Taking derivatives to get accelerations we have

$$\frac{d^2x}{dt^2} = \frac{d^2u}{dt^2} + L\frac{d^2\theta}{dt^2}\cos\theta(t) - L(\frac{d\theta}{dt})^2\cos\theta(t)$$

and

$$\frac{d^2y}{dt^2} = -L\frac{d^2\theta}{dt^2}\sin\theta(t) - L(\frac{d\theta}{dt})^2\cos\theta(t)$$

Now we do force balance through the rod where $F(t)$ is the force applied by the rod in the radial directiona. In the x direction we have

$$F(t)\sin\theta(t) = M\frac{d^2x}{dt^2} = M\left(\frac{d^2u}{dt^2} + L\frac{d^2\theta}{dt^2}\cos\theta(t) - L(\frac{d\theta}{dt})^2\sin\theta(t)\right)$$

Likewise in the y direction we have

$$-Mg + F(t)\cos\theta(t) = m\frac{d^2y}{dt^2} = M\left(-L\frac{d^2\theta}{dt^2}\sin\theta(t) - L(\frac{d\theta}{dt})^2\cos\theta(t)\right)$$

Solving each equation for $F(t)$, equating and then a bit of algebra/trigonometry yields the given relation.

(b) *(10 points)* Rutgera has available to her power software which only works on statespace representations of systems. Linearize the system about $\theta = 0$ (vertical pendulum) and write out the statespace form of the linearized system.

Soln: Almost a no-brainer here. $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ for small θ so we have (after getting rid of the extraneous L's and M

$$g\theta(t) = L\frac{d^2\theta}{dt^2} + \frac{d^2u}{dt^2}$$

Use θ and $\dot{\theta}$ as the state variables and we have

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix}\boldsymbol{\theta} - \begin{bmatrix} 0 \\ 1/L \end{bmatrix}\frac{d^2u}{dt^2}$$

(c) *(10 points)* From your linearized result, show that a small perturbation away from exactly vertical will cause a rapid deviation of the pendulum from vertical. Don't just say it's obvious that the pendulum will fall down. Show this result from your linearized equations.

Soln: The solution will contain terms of the form $e^{\lambda t}$ where the λ are roots of $\lambda^2 - (g/L)^2$. One of these roots is positive if $g \neq 0$. Thus, if the mass is displaced even the slighested BIT away from vertical (the equilibrium point), the solution will grow exponentially away from $\theta = 0$ and the linearization breaks down.

(d) *(10 points)* If the input is $u(t)$ and the observed output is $\theta(t)$, please derive a transfer function for this linearized system.

Soln: Let $\frac{d^2u}{dt^2} = f(t)$ with Laplace transform $F(s)$. Our output is $\theta(t)$ so $y(t) = C\boldsymbol{\theta}$ where $C = [1, 0]$ We then have

$$H(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ -g/L & s \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1/L \end{bmatrix}$$

Doing the linear algebra yields

$$H(s) = \frac{1}{s^2 - g/L}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ g/L & s \end{bmatrix}\begin{bmatrix} 0 \\ -1/L \end{bmatrix} = \frac{-1}{Ls^2 - g}$$

From $u(t)$ to $y(t) = \theta(t)$ we then have

$$H(s) = -\frac{s^2 - su(0) - \dot{u}(0)}{Ls^2 - g}$$

One could also derive this directly from the scalar linearized equation.

(e) *(10 points)* Rutgera has been charged to devise a feedback system which keeps the pendulum bob in a vertical position. She has decided to make

$$\frac{d^2u}{dt^2} = a\theta(t) + b\dot{\theta}(t)$$

where a and b are real constants.

What values of a and b will cause the pendulum bob to be restored to an upright position if it is perturbed slightly from vertical?

Soln: We can go back to the original equation and see that now

$$MgL\sin\theta(t) = ML^2\frac{\partial^2\theta}{\partial t^2} + (a\theta(t) + b\frac{d\theta}{dt})ML\cos\theta(t)$$

Linearizing we have

$$g\theta(t) = L\frac{\partial^2\theta}{\partial t^2} + (a\theta(t) + b\frac{d\theta}{dt})$$

Rearranging

$$\frac{\partial^2\theta}{\partial t^2} = \frac{g-a}{L}\theta - \frac{b}{L}\frac{d\theta}{dt}$$

Note that we could also have stayed in statespace form if we'd wanted to.

The problem before was that at least one of the eigenvalues was positive and this led to exponential growth of θ. What we need for the bob to be restored is to have NEGATIVE REAL PARTS to all eigenvalues.

Looking at the characteristic equation we have $s^2 + \frac{b}{L} - \frac{g-a}{L} = 0$ so the roots are

$$s = \frac{1}{2}\left[-\frac{b}{L} \pm \sqrt{\frac{b^2}{L^2} + 4\frac{g-a}{L}}\right]$$

It should be reasonably clear that b must be positive and that we must have $\sqrt{\frac{b^2}{L^2} + 4\frac{g-a}{L}} < \frac{b}{L}$. This can only happen if $a > g$.

Watching this sort of system in operation is pretty neat. You can trade off response time (making b and a really large) against bugaboos which occur to make the system even more unstable under feedback (delays in measuring θ and its derivative).

(f) *(10 points)* Does your answer to the previous part change if the pendulum is started off the downward vertical position ($\theta = \pi$)? Think carefully.

Soln: The basic idea here is that for a and b large enough, once you get the pendulum bob near vertical, it'll stick there. The trick is getting it there. So we have to use heuristics to see if the feedback we provide will make the system rock the mass into an upright position.

So, once again, consider the new plant equations:

$$L\frac{\partial^2\theta}{\partial t^2} = g\sin\theta(t) - (a\theta(t) + b\frac{d\theta}{dt})\cos\theta(t)$$

We have to see if the dynamics are such that the pendulum is driven toward upright vertical whatever it's state happens to be. That is, we must evaluate the sign of $g\sin\theta(t) - (a\theta(t) + b\frac{d\theta}{dt})\cos\theta(t)$ and make we get the right restorative forces.

The first observation is that a and b should be pretty large so that the effect of the term in g is essentially negligible except for θ in very small neighborhoods around $\pi/2$ and

$-\pi/2$ where acceleration in u exerts no torque about the pivot point. Since the bob falls out of these neighborhoods naturally, we can safely ignore the term in g.

So now we have

$$L\frac{\partial^2\theta}{\partial t^2} = -(a\theta(t) + b\frac{d\theta}{dt})\cos\theta(t)$$

Since the state of the system is completely determined by θ and $\dot{\theta}$ let's examine the forces for various combinations of θ and $\dot{\theta}$. Let's number the quadrants where the pendulum can reside as I, II, III and IV clockwise starting from the upper right where $0 \leq \theta \leq \pi/2$. Then consider the direction of motion of the pendulum bob (positive $\dot{\theta}$ is clockwise). What we want to know is the direction of acceleration (being applied by our control system) when the bob is in various combinations of such "states". The entries in the following table state in loose terms the effects of a and b by looking at their signs.

$\dot{\theta}$	I	II	III	IV
+	$-(\tilde{a}+\tilde{b})$	$(\tilde{a}+\tilde{b})$	$-(\tilde{a}+\tilde{b})$	$(\tilde{a}+\tilde{b})$
−	$-(\tilde{a}-\tilde{b})$	$(\tilde{a}-\tilde{b})$	$-(\tilde{a}-\tilde{b})$	$(\tilde{a}-\tilde{b})$

Table 1: Pendulum control acceleration

We see that we get exactly the restorative force we'd like if $\tilde{a}$ dominates $\tilde{b}$ in quandrants I and IV. So for reasonably small velocities in quandrants I and IV, the pendulum will be restored to the vertical.

In quandrants II and III we see that for small velocities, an oscillation is set up (like a child on a swing). Since a and b are assumed large, this rocking eventually increases the rotational velocity until the pendulum swings into either quandrant I or IV. At that point it is "captured" by the dynamics of the system for $-\pi/2 < \theta < \pi/2$ and restored to the vertical.

ASIDE: This problem would make a nice numerical exercise or even a nice lab project! What I imagine is a rod mounted to a motor. The motor rotates the rod like a helicopter blade. At the end of the rod is a pendulum whose motion is normal to the rod. This does away with the need for a really long track in the u direction. However, The pendulum and rod combination would have to be sturdy so that no centrifugal forces and resultant stresses would muck things up.

Needless to say, the real dynamics are a might more complicated owing to delays in measuring things. If you're lucky, this manifests itself as filters through which "pure" variables must pass before actually being observed for continuous time systems. If the system to be controlled is either far enough away from the controller or operates fast enough that the speed of light is not all that quick, then you have REAL HONEST TO GOODNESS delays and THESE ARE BEARS TO DEAL WITH because they introduce $e^{-\tau s}$ terms into the transfer functions of continuous time systems. Analytically however, it's not as problematic in discrete time systems.

2. *(50 points)* **Some Simple Proofs To Keep You Sharp**

   Please prove/disprove the following statements:

   (a) *(10 points)* If $x(t)$ has Laplace transform $X(s)$, then dx/dt has transform $sX(s) - x(0)$
   Soln: Integrate $\int_0^\infty \dot{x}(t)e^{-st}dt$ by parts and the result pops out.

   (b) *(10 points)* If $x(t)$ has Laplace transform $X(s)$, then $\int_0^t x(r)dr$ has transform $X(s)/s$.
   Soln: Interchanging the order of integrations makes me queasy, but you can do that here to get the result. That is, integrate both sides of $x(t) = L^{-1}\{X(s)\}$ from 0 to t and the

result pops out. You may ask why you could not simply differentiate both sides in the previous part. The reason is that the definite integral takes care of any constants lying about whereas the differentiation ignores (possibly fatally) constants. For example, by blundering along using simple differentiation you could "prove" that $x(t) + a$ and $x(t)$ have the same Laplace transform (WRONG!!!). One could also use integration by parts if desired, or doctor the result of the previous part.

(c) *(10 points)* e^{At} is always invertible regardless of whether A is invertible.

Soln: The most straightforward way is to expand in a taylor series and show that $e^{At}e^{-At} = I$. There is some algebraic bookeeping involved but I'll leave the details to you. Another even simpler way is to invoke the sanctity of transition matrices which are ALWAYS invertible. That is, $phi(t, t_0) = phi(t_0, t)^{-1}$. Since $phi(t, t_0) = e^{A(t-t_0)}$ for a system $\dot{x} = Ax$, you're done.

(d) *(10 points)* The Laplace transform of e^{At} is $(sI - A)^{-1}$.

Soln: Take Laplace transform of $\dot{x} = Ax$ to obtain $sX(s) - x(0) = AX(s)$. Rearrange to get $X(s) = (sI - A)^{-1}x(0)$. Take the inverse Laplace of both sides and the result falls out. You cannot just use scalar exponential properties unless you state why they can be used. Specifically, $e^A e^B \neq e^{A+B}$ unless A and B commute.

(e) *(10 points)* The Laplace transform of $t^n e^{at} = n!/(s-a)^{n+1}$.

Soln: First remember that $L[e^{at}] = 1/(s-a)$. Then repeatedly (n times) differentiate both sides with respect to a to obtain the result. I was amazed at the varied and complicated ways people tried to prove this result. Some intrepid souls used combinations of properties of Laplace transforms (rather elegant I thought).

3. *(60 points)* **Obligatory Hard Problem**

You are given a system with statespace description

$$\dot{x} = A(t)x$$

Please derive an expression for the transition matrix of this system in terms of the exponential function. You must state precisely when your expression is valid in terms of operations on the matrix $A(t)$. You may assume that $A(t)$ is a completely continuous matrix function.

Good luck! This is hard.

Soln:

First, the answer is:

$$\phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau}$$

PROVIDED THAT $A(t)\int_{t_0}^{t} A(\tau)d\tau = \int_{t_0}^{t} A(\tau)d\tau A(t)$; that is provided $A(t)$ and its integral commute. Pretty obvious that if $A(t) = A$ we end up with the standard $e^{A(t-t_0)}$.

You could prove this a few ways. The first is the MACHO!/MACHETTE! way using contraction maps which essentially starts from scatch. This solution is completely general but maybe a bit useless from a human analyst's standpoint unless you see that under the commutation condition it is identical to the simple answer given above.

So, from our contraction mapping theory we have

$$\begin{aligned}\phi(t, t_0) &= \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\tau_1)d\tau_1 + \int_{t_0}^{t}\int_{t_0}^{\tau_1} \mathbf{A}(\tau_1)\mathbf{A}(\tau_2)d\tau_2 d\tau_1 \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2} \mathbf{A}(\tau_1)\mathbf{A}(\tau_2)\mathbf{A}(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \cdots\end{aligned}$$

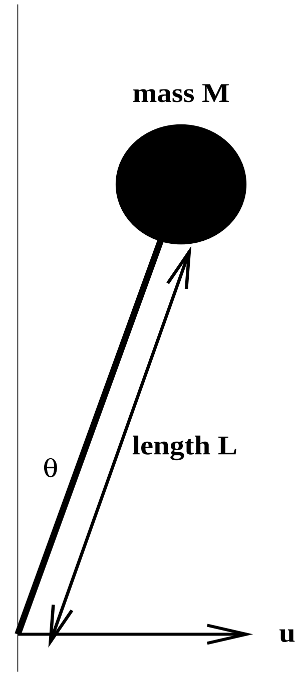

Figure 1: Inverted pendulum

Which is valid for all $\mathbf{A}(t)$ such that $\mathbf{f}(\mathbf{x}(t), t) = \mathbf{A}(t)\mathbf{x}(t)$ satisfies the Lipschitz condition. That is, this is ALWAYS valid! If you provided this result, you did not have to provide the exponential form (although it kind of suggests itself when you differentiate the function w.r.t. t.)

The second method is by guessing the answer and verifying it. This is easy if you remember the differentiation properties for matrix exponentials and taylor form of the exponential function (for the commutation part).

NOTE: There are probably other methods of solution as well. Also, I was DISMAYED by the number of people who tried to use variable separability WITH VECTORS and MATRICS. Remember, you cannot divide by a vector or matix!