Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 Systems Analysis Fall 1997

FIRST EXAMINATION STATEMENT & SOLUTIONS

You have THREE WHOLE HOURS to answer the following three (3) questions. The point values are as shown. You are allowed two sides of an 8.5 X 11 sheet of notes for reference. The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT.

GOOD LUCK!

1. *(40 points)* **Contraction Maps and Matrices**

   This problem is almost identical to one which appeared on last year's midterm. Why so easy? Because I want you to CALM DOWN and work through something you know for the first problem. All vectors and matrices are assumed to be elements of $\Re^N$ and $\Re^N \times \Re^N$ respectively.

   (a) *(10 points)* We can state with certainty that the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is a contraction if:

      i. $||\mathbf{A}||_1 < 1$
      ii. $||\mathbf{A}||_2 < 1$
      iii. $||\mathbf{A}||_\infty < 1$
      iv. (a) or (b) or (c) [logical or]
      v. None of the above

   (**no partial credit on this part**).

   If ANY matrix norm is less than unity, then the mapping is a contraction. The correct answer is (d).

   (b) *(10 points)* Suppose for some $\mathbf{x}^* \neq \underline{0}$ we have $\mathbf{x}^* = \mathbf{A}\mathbf{x}^*$. Can the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ be a contraction over $\Re^N$? Why/why not?

   A contraction mapping has a unique fixed point. Since zero is always a fixed point of this mapping, a nonzero fixed point implies non-uniqueness. Thus the mapping cannot be a contraction. Another way to go was to remember that $|\lambda| \leq ||A||$ WHATEVER norm is used. Since $Ax = x$ implies an eigenvalue of 1, the smallest ANY norm could be is $||A|| = 1$ and that's not small enough for a contraction map.

   (c) *(10 points)* We know that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ where $\{\lambda_i\}$, $i = 1, 2, ...N$ are the eigenvalues of matrix $\mathbf{A}$ and $\{\mathbf{v}_i\}$ are the associated eigenvectors. For this part of the problem we assume that the λ_i are distinct so that the $\mathbf{v}_i$ are linearly independent and any vector $\mathbf{x}_0 \in \Re^N$ can be written as

   $$\mathbf{x}_0 = \sum_{i=1}^{N} c_i \mathbf{v}_i$$

with a suitable choice of the constants $\{c_i\}$.

Please derive an expression for $\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0$ and **prove** that

$$\lim_{k \to \infty} \mathbf{x}_k = 0$$

for all $\mathbf{x}_0 \in \Re^N$ **iff** $|\lambda_i| < 1$, $i = 1, 2, \ldots N$.

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 = \mathbf{A}^k \sum_{i=1}^{N} c_i \mathbf{v}_i = \sum_{i=1}^{N} \lambda_i^k c_i \mathbf{v}_i$$

Now

$$0 \le || \sum_{i=1}^{N} \lambda_i^k c_i \mathbf{v}_i || \le \sum_{i=1}^{N} |\lambda_i^k| || c_i \mathbf{v}_i ||$$

For arbitrary v_i and c_i, if any of the λ_i have magnitude greater than or equal to 1, the sum cannot go to zero in k. If all the $|\lambda_i| < 1$ then each term goes to zero in k.

(d) *(10 points)* Well, looky there! The mapping in the previous part always converged to a fixed point (of zero). Is the condition on the eigenvalues stated in part (1c) necessary and sufficient for the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ to be a contraction for all matrices $\mathbf{A}$ with distinct eigenvalues in all metric spaces?

Prove or find a counterexample.

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 100 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The magnitudes of the two eigenvalues (along the diagonal of this upper-triangular matrix) are both 1/2. However, both the ρ_1 and ρ_∞ norms are MUCH larger than 1. So $|\lambda_i| < 1 \ \forall i$ is necessary but INSUFFICIENT to guarantee a contraction mapping.

2. *(40 points)* **More Maps**

(a) *(10 points)* When, if at all, are the following mappings convergent? Be as complete as possible in the time you have. The more complete the answer the closer to full credit you will receive.

i. *(5 points)* $x_{n+1} = ax_n + b$ for $a, b, x_n \in \Re$

This map is just a difference equation with homogeneous solution $x_n^h = Ca^n$ with C a constant. The particular solution is $x_n^p = b/(1-a)$. As long as $|a| < 1$ we'll have a convergent series.

ii. *(5 points)* $x_{n+1} = a(1 - x_n)^2$ for $x_n \in [0, a]$, $a \ge 0$.

This map is a pain in the buttocks and has driven many a student and researcher close to insanity. For some values of a, the map veers about wildly. For others it's pretty boring and for others it has some finite number of fixed points.

I was looking to see how you play in the face of the unknown on this one. The course meat is in the next part. That is I accepted almost anything (mathematically correct of course) for this part as an answer. For example, the mapping is convergent for $a = 0$ true (and boring). It's also convergent for $a = 0.25$. However, it will also bust out of the interval specified if $a > 2$.

Hope you had fun! This part had no single answer.

(b) *(10 points)* When, if at all, are the above mappings contractions (you may assume a metric space with $\rho_1()$ as the metric.

For the linear map we have

$$|a(x_1 - x_2)| = |a||x_1 - x_2| = |a|\rho_1(x_1, x_2)$$

So as long as $|a| < 1$ *we've got a contraction. BUT DON'T GET FOOLED INTO THINKING THAT as long as the roots of the char poly are inside the unit circle, you have a contraction.*

For the next (chaotic) mapping we have,

$$\begin{aligned}
|a(1-x_1)^2 - a(1-x_2)^2| &= |a||(1-x_1)^2 - (1-x_2)^2| && (1)\\
&= |a||2(x_2 - x_1) - (x_2 - x_1)(x_1 + x_2)| && (2)\\
&= |a||(2 - x_1 - x_2)||(x_2 - x_1)| && (3)\\
&= |a||(2 - x_1 - x_2)|\rho(x_2, x_1) && (4)\\
& && (5)
\end{aligned}$$

The largest magnitude $2 - x_1 - x_2$ *can be is* $2|1-a|$ *or* 2 *(whichever is larger). So we need* $|a|max(2|1-a|, 2) < 1$*. This translates to* $a < 1/2$ *for the mapping to be a contraction (remember* $a \geq 0$ *in the problem definition).*

(c) *(10 points)* PROVE whether or not the following mappings are bounded

i. *(5 points)* $A(x) = x^2$ for $x \in [0, 1]$
On $[0, 1]$ *we have* $dA(x)/dx \leq 2$ *so the mapping is bounded. The exact proof just uses the Mean Value theorem as before.*

ii. *(5 points)* $A(x) = \sqrt{x}$ for $x \in [0, 1]$
On $[0, 1]$ *we have* $dA(x)/dx$ *unbounded. So to prove unboundedness consider the two points* $x_1 = 0$ *and* $x_2 = \epsilon$ *where* ϵ *is a small positive constant. Form the ratio of* $|A(x_2) - A(x_2)|$ *to* $|x_2 - x_1|$ *as*

$$\frac{\sqrt{x_2} - \sqrt{x_1}}{x_2 - x_1} = \sqrt{\frac{1}{\epsilon}}$$

and we see it's unbounded since ϵ *can be arbitrarily small. Therefore the mapping is unbounded.*

(d) *(10 points)* Using contraction mapping theory, prove that the equation $xe^x = 1/2$ has a unique solution x^* and provide a method to iteratively calculate x^*.

We have by rearrangement $x = e^{-x}/2$*. So our mapping function is* $f(x) = e^{-x}/2$*. But is it a contraction? Well, don't we need to specify a space? Let's try* $x \geq 0$ *for starters. First we see that the range of* $f(x)$ *is* $x \geq 0$ *so we're guaranteed that* $f(x)$ *maps back into the space we've chosen. Now, since* e^{-x} *is continuous, we can use our favorite book trick (mean value theorem) and note that* $|d(e^{-x})/dx| = |-e^{-x}| \leq 1\ \forall x \geq 0$*. So...*

$$\begin{aligned}
|f(x_1) - f(x_2)| &= \frac{1}{2}|e^{-x_1} - e^{-x_2}| && (6)\\
&= \frac{1}{2}|x_1 - x_2|e^{\xi} && (7)\\
&\leq \frac{1}{2}|x_1 - x_2| && (8)\\
&< |x_1 - x_2| && (9)
\end{aligned}$$

and we're done proving it's a contraction.

We then form $x_{n+1} = \frac{1}{2}e^{-x_n}$ *and it's guaranteed to converge to a unique point* x^* *in the space we've chosen. This is a neato trick which we'll use again and again. AND you might even find it useful yourself for some things.*

NOTE: you might want to think about what might have happened if we'd insisted that our space be $\Re$ instead of the non-negative reals. Does the limit still exist? (it does). Is the mapping still a contraction? (it's not).

3. *(40 points)* **Fun With Differential Equations**

   Find the unique solution (if it exists) to the following initial value problems for $t \geq 0$ USING CONTRACTION MAPPINGS.

   Let me tell you a story I heard. In a medical school class somewhere they were discussing renal failure and diabetes. The instructor brought a beaker of urine with him and told the class that you could detect diabetes by sugar in the urine. He then quickly dipped a finger into the beaker and stuck it into his mouth. Then he made each student (about 100 of them) come down and repeat the experiment. Some claimed they could taste the sugar, and others did not.

   After everyone was done, he carefully explained that you cannot taste the sugar in urine and redid the experiment for the class more slowly. He dipped his MIDDLE finger in the beaker but put his INDEX finger in his mouth. The purpose of the exercise, he then explained, was to make these future doctors remember to be observant and don't take things at face value!

   Well, if you liken trying to solve difficult integrals to manually testing for sugar in urine then I've passed the lesson on to you.

   Part (a) of this problem is easy since we've done it before. However, if you blindly try the next three you get stung. The trick is to realize WHEN you can apply your lovely machinery. Since no range was given, you have to assume that $x(t) \in \Re$ and look carefully at $\dot{x} = f(x,t)$. For all but the first part, $f(x,t)$ IS NOT BOUNDED! So you can't just blindly apply the contraction mapping theory. Mind you, just because our contraction mapping machinery won't work over the reals DOES NOT mean that no solution exists. It just means that our iteration method is not guaranteed to work over all the reals. If you carefully specify intervals, the last two can be (and were by some of you) solved by the iteration method. The second problem is a lost cause unless you are the (Gauss + Liebnitz)^2 of integrators.

   So what I wanted you to say is "not bounded so can't iterate."

   Those of you who tried to evaluate those integrals learned a lesson, no? But I did not penalize agonizing effort. You only got docked a 5 points if you did not specify the space over which the contraction map had to be valid.

   ASIDE: If you restrict the intervals, however, and have a computer handy, this method can probably still be used in a practical sense. What you might do is compute the integral numerically to some precision and iterate on the values obtained. Maybe we'll do that for a homework problem. Or better yet, maybe one of you will try it for part (b) of this problem and post snapshots of your results on the web. I'll link the page if you do and you'll be 501-FAMOUS! What I'd expect (and love) to see is an animation of a starting squiggle (user entered?) which after iteration starts looking like $\sqrt{2t+1}$.

   (a) *(10 points)* $\dot{x} = x$ with $x(0) = 1$

   This is our old friend from class, e^x. Notice that $f(x) = x$ is bounded everywhere in $\Re$. We've done it in class and it's in the book so I won't do it again here.

   (b) *(10 points)* $\dot{x} = \frac{1}{x}$ with $x(0) = 1$

   The function $1/x$ is unbounded and therefore does not satisfy the Lipschitz condition in general. If you went on and tried to solve this beast using the contraction mapping method then you found a new definition of pain and agony.

   However, a solution DOES exist (if you restrict the domain/range) and can be found easily by variable separability as $x(t) = \sqrt{2t+1}$. And this solution IS a fixed point of *the mapping:*

$$x_1(t) = 1 + \int_0^t \frac{1}{\sqrt{2t+1}} dt = 1 + (\sqrt{2t+1} - 1) = \sqrt{2t+1}$$

(c) *(10 points)* $\dot{x} = -x^2$ with $x(0) = 1$

Not bounded over reals so no go for iteration method in general. However, a solution does exist (you just can't say it can be found using our iteration method): $x(t) = \frac{1}{t+1}$. *The answer you would have gotten if you kept going with the integrals would have been the series representation* ($\sum_n (-t)^n$).

If you tried the iterations, you might have actually gotten somewhere, but again, it's hard. As before, the solution is a fixed point of the mapping.

(d) *(10 points)* $\dot{x} = x^2$ with $x(0) = 1$ *Not bounded again! But again, solution exists on a restricted interval,* $x(t) = \frac{1}{1-t}$, *and is a fixed point of the mapping. Here too you might have gotten close if you were persistent* ($\sum_n (t)^n$).