Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 Systems Analysis

Problem Set 8 SOLUTIONS

S & B: Ch.[5], prob. 1:
The transition matrix can be found in P3-1. We can form the controllability Gramian:

$$W(t_0,t_1)=\int_{t_0}^{t_1}\phi(t_0,\tau)B(\tau)B^T(\tau)\phi^T(t_0,\tau)d\tau=(t_0-t_0^2/t_1)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

Therefore, $|W(t_0,t_1)|=0$, the system is not controllable. **QED**

S & B: Ch.[5], prob. 2:
We have found the transition matrix in P1-16:

$$\phi(t,t_0)=\begin{bmatrix}\frac{2}{3}+\frac{1}{3}e^{3(t-t_0)} & \frac{-1}{3}+\frac{1}{3}e^{3(t-t_0)}\\ \frac{-2}{3}+\frac{2}{3}e^{3(t-t_0)} & \frac{1}{3}+\frac{2}{3}e^{3(t-t_0)}\end{bmatrix}$$

From the controllability Gramian, we have

$$W(t_0,t_1)=\int_{t_0}^{t_1}\phi(t_0,\tau)B(\tau)B^T(\tau)\phi^T(t_0,\tau)d\tau$$

Therefore

$$W(0,1)=\begin{bmatrix}\frac{11}{18}+\frac{-4}{27}e^{-3}+\frac{-1}{54}e^{-6} & \frac{-1}{3}+\frac{-2}{27}e^{-3}+\frac{-1}{27}e^{-6}\\ \frac{-1}{3}+\frac{-2}{27}e^{-3}+\frac{-1}{27}e^{-6} & \frac{2}{9}+\frac{8}{27}e^{-3}+\frac{-2}{27}e^{-6}\end{bmatrix}$$

$$|W(0,1)|=\frac{2}{81}+\frac{8}{81}e^{-3}-\frac{2}{27}e^{-6}-\frac{4}{81}(e^{-3})^2=0.293$$

which is nonsingular, thus, the system is completely controllable in [0,1]. **QED**

S & B: Ch.[5], prob. 3:
We have found $W(0,1)$ in P5-2. (i). Obviously, it is symmetric. (ii). $\triangle_0=\frac{11}{18}+\frac{-4}{27}e^{-3}+\frac{-1}{54}e^{-6}=0.603>0, \triangle_1=0.293>0$, therefore $W(0,1)$ is positive definite. **QED**

S & B: Ch.[5], prob. 4:

$$K=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$$

and $|K|=2$, therefore K is full rank, and the system is completely controllable. **QED**

S & B: Ch.[5], prob. 8:
This problem is a little tricky. We can verify the system is completely controllable by forming the $\mathbf{K}$ matrix

$$\mathbf{K}=[\mathbf{B},\mathbf{AB}]=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$$

which has full rank. So it would seem that we should be able to force the system state along any trajectory we like. However, equation 5.19 states $\mathbf{X}(s)=\mathbf{R}(s)\mathbf{X}_0+\mathbf{R}(s)\mathbf{B}\mathbf{U}(s)$, where

$$\mathbf{R}(s)=(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{(s-1)(s-2)-2}\begin{bmatrix}s-2 & 1\\ 2 & s-1\end{bmatrix}$$

therefore

$$\begin{bmatrix}\frac{1}{s^2}+\frac{1}{s}\\ \frac{1}{s}\end{bmatrix}=\frac{1}{(s-1)(s-2)-2}\begin{bmatrix}s-1\\ s+1\end{bmatrix}+\frac{1}{(s-1)(s-2)-2}\begin{bmatrix}s-2\\ 2\end{bmatrix}\mathbf{U}(s)$$

From the first row of the above equation, we have $U(s) = -\frac{s+3}{s(s-2)}$ From the second row we have $U(s) = -2$. Thus, there is no consistent solution!

But the system was completely controllable so HOW CAN THIS POSSIBLY BE!!!????!!!???? Think a bit about it. We'll explore exactly this type of problem on the final to resolve this seeming paradox.

S & B: Ch.[5], prob. 10:
We have found the state space representation

$$\dot{X}(t) = -\frac{R_2 + R_1}{L}X(t) + \frac{1}{L}U(t)$$

therefore it's controllability Gramian $K = \frac{1}{L}$ is not zero, so the system is completely controllable. **QED**

7. Part (a)
We can figure out

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ -p_0 & -p_1 & -p_2 & \dots & -p_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix}$$

the controllability matrix:

$$K = \begin{bmatrix} B, & AB, & \dots, & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 & -p_{n-1} \\ . & . & . & . & . \\ 0 & 1 & -p_{n-1} & \dots & -p_2 \\ 1 & -p_{n-1} & \dots & -p_2 & -p_1 \end{bmatrix}$$

which is obviously full rank.

Part b.

The easiest thing is to approach the problem in almost complete ignorance. In fact, I highly recommend ignorance when approaching a new problem. Thinking about it independently first has two effects: 1) you might discover an new route to solution, 2) you'll better appreciate and understand someone else's solution once you really understand the problem.

First does the system satisfy Lipschitz? Yes it does: $|x_1^2 - x_2^2| \le L|x_1 - x_2|$ for $x_1, x_2 \in X$ where X is some finite neighborhood about any point (say point a).

Now consider that for ANY given trajectory $x(t)$ we obtain the necessary drive term $u(t)$ as $u(t) = \dot{x} - x^2$. Thus we can easily derive a control to take us along any trajectory we are given.

So, suppose we want to get from x_0 at t_0 to x_1 at t_1. We simply form the trajectory $x(t) = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0) + x_0$ which takes us from x_0 at t_0 to x_1 at t_1. We then find

$$u(t) = \frac{x_1 - x_0}{t_1 - t_0} - (\frac{x_1 - x_0}{t_1 - t_0}(t - t_0) + x_0)^2$$

Yup, this baby is completely controllable!

8. TURKEY PROBLEM
As Nike says "just do it!". The solution to the difference equation with $w(0) = 1$ is $w(10) = \prod_{t=0}^{9}(2 + u(t))$. Now the question becomes how do you maximize that product term?

Consider the product $2(2 + k)$. For $k > 1$ we always have $3(1 + k) > 2(2 + k)$. Now notice that if some $u_i > 1$ then some other $u_j = 0$. Thus, if more than one unit is given in any week, the product can always be increased by using the extra unit from one week in the week where no units were used.

This argument leads to a uniform distribution on additive units, 1 per week. The MAXIMUM TURKEY WEIGHT is then $w(10) = 3^{10} = 55,049$. So let's hope that the turkey was an EMBRYO when the process started!