Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 Systems Analysis

Problem Set 7 SOLUTIONS

S & B: Ch.[4], prob. 13:
We use norms and the multivariate mean value theorem from chapter 1.

$|x_1(t) - x_2(t)| = |f(x_1) - f(x_2)| = |f'(\xi)| |x_1 - x_2|$ for some ξ. Since $|f'(\xi)|_2 < 1$, the mapping is a contraction mapping. Thus, it always converges to the fixed point, which is 0. Therefore the sequence converges to zero.

S & B: Ch.[4], prob. 14:

- $$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & 1 \\ -1 & \lambda + 4 \end{vmatrix}$$

  which yields
  $$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0, \quad \lambda_1 = \lambda_2 = -3.$$
  Thus the eigenvalues of A have negative real parts.

- Assume
  $$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$$
  $$A^T Q + QA = \begin{bmatrix} -4q_{11} + 2q_{12} & -q_{11} - 6q_{12} + q_{22} \\ -q_{11} - 6q_{12} + q_{22} & -2q_{12} - 8q_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
  which means, $-4q_{11} + 2q_{22} = -1$, $-q_{11} - 6q_{12} + q_{22} = 0$ and $-2q_{12} - 8q_{22} = -1$. We can solve the equations to obtain $q_{11} = \frac{13}{54}$, $q_{12} = -\frac{1}{54}$ and $q_{22} = \frac{7}{54}$.

S & B: Ch.[4], prob. 15:
The characteristic equation is

$$\varphi(\lambda) = \lambda^4 + 2\lambda^3 + 3\lambda^2 + 4\lambda + 5$$

The roots of the equation can be found (using a root finder) as:

$$\lambda_1 = 0.2878 + i1.416, \quad \lambda_2 = 0.2878 - i1.416$$

$$\lambda_3 = -1.2878 + i0.8579, \quad \lambda_4 = -1.2878 - i0.8579$$

Therefore it has two roots on the RHS plane, the system is unstable. We can also determine the location of roots with the Routh Array method.

S & B: Ch.[4], prob. 16:
We can verify that K is positive definite by $\delta_1 = 4 > 0$, $\delta_2 = 15 > 0$. We can also verify that

$A+A^T$ is negative definite by verifying that $-(A+A^T) = \begin{bmatrix} -8 & 3 \\ 3 & -12 \end{bmatrix}$ is postive definite;i.e., $\delta_1 = 8 > 0,\ \ \delta_2 = 87 > 0$. The eigenevalues of KA can be found as $\lambda_1 = -15,\ \lambda_2 = -22$. It can be seen that the eigenvalues of KA are all negative, which is stated in Theorem 4.9.

S & B: Ch.[4], prob. 17:

$$\varphi(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

The Routh Array is formed as

$$\begin{array}{llll} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & 1 & 5 & \\ s^1 & -6 & 0 & \\ s^0 & 5 & & \end{array}$$

The sign changes twice in the first column, thus the system has two roots with postive real part.[1] Therefore the system is unstable.

S & B: Ch.[4], prob. 18:
and **S & B:** Ch.[4], prob. 19:

If the eigenvalues of the matrices are in the LHP then we have BIBO. So we have characteristic equations of

$$s^2 + 37s + 330$$

and

$$s^2 + 10s + 22$$

Hurwitz gives δ_1 and δ_2 greater than zero for both so that all poles lie in the LHP. This implies assymptotic stability and as a consequence for LTI systems, BIBO stability.

S & B: Ch.[4], prob. 21:

$$\varphi(\lambda) = det(A - \lambda I) = det \begin{bmatrix} -\frac{R_f}{L_f} - \lambda & \frac{K_m K_a K_f}{L_f} & 0 \\ \frac{1}{J} & -\frac{f}{J} - \lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}$$

which yields

$$-(\lambda^3 + (\frac{R_f}{L_f} + \frac{f}{J})\lambda^2 + \frac{fR_f - K_m K_a K_f}{JL_f})$$

Let $b = \frac{R_f}{L_f} + \frac{f}{J}$ and $c = \frac{fR_f - K_m K_a K_f}{JL_f}$, we can find the eigenvalues of the equation as

$$\lambda_1 = 0,\ \lambda_2 = \frac{-b + \sqrt{b^2 - 4c}}{2},\ \lambda_3 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Therefore the system is stable as long as $c > 0$. When $c = 0$ we get repeated zero eigenvalues and it turns out that we are short one eigenvector. Thus, the system is not stable for $c = 0$.

[1] We did not cover this last bit about the NUMBER of positive real roots so you can safely ignore it. However if you're interested, check out the section on Routh arrays in C.T. Chen's book on linear systems