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330:501 Systems Analysis

Problem Set 5 SOLUTIONS

S & B: Ch.[3], prob. 13:
Assuming zero initial conditions and taking the Laplace transform, we can get the transfer function as following:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{R_2}{sL + R_1 + R_2}$$

Chosing $x(t) = i(t)$, we can find the state space representation and its solution:

$$\dot{x}(t) = -\frac{R_2 + R_1}{L}x(t) + \frac{1}{L}u(t)$$

$$y(t) = R_2 x(t)$$

So that

$$x(t) = e^{-\frac{R_1+R_2}{L}t}x(0) + \frac{1}{L}\int_0^t e^{-\frac{R_1+R_2}{L}(t-\tau)}u(\tau)d(\tau)$$

If $u(t) = 1$, the equilibrium is found as $x(t) = \frac{1}{R_1+R_2})$ which makes $y(t) = \frac{R_2}{R_1+R_2}$, a classical voltage divider.

S & B: Ch.[3], prob. 15:
Applying Theorem 3.5, we have

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(t)$$

Could find transition matrix but not worth it except for practice.

S & B: Ch.[3], prob. 17:
Assuming zero initial conditions and taking the Laplace transform, we have the transfer function as following:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+5s+6}$$

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S & B: Ch.[3], prob. 20:
The transfer function is:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2(s-1)}{(s+1)^2[s^2(s-1) - 2s + 1]}$$

S & B: Ch.[3], prob. 21:
The overall transfer function is

$$H(s) = \frac{\Theta(s)}{V(s)} = \frac{K_m K_a}{s[(sL_f + R_f)(sJ + f) + K_m K_a K_t]} = \frac{b_0}{s^3 + a_2 s^2 + a_1 s}$$

where $b_0 = \frac{K_m K_a}{L_f}$, $a_2 = \frac{L_f f + R_f J}{L_f}$ and $a_1 = \frac{f R_f + K_m K_a K_t}{L_f}$. From Theorem 3.5, we obtain,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_1 & -a_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_0 & 0 & 0 \end{bmatrix}$$

The state variables of this realization are artificial in that no care was taken to have them correspond to real physical variables. One could define state variables directly corresponding to the inductance current, the inertia. Feedback adds another state variable (not in general of course, but usually). However, all realizations which are *similar* to the above system are equivalent.

Now as for the output when the input is 1, That corresponds to a DC input ($s = 0$) for all time. Thus, since $\omega(t) = \frac{1}{K_t}$ we'd have $\theta(t) = t/K_t + \theta_0$. If, however, we have the input as a unit step, then the solution is more complicated and we'd need the roots of the characteristic equation to proceed exactly

However, since ω is second order, use the overdamped, underdamped, critically damped arguments to obtain a step response for ω similar to figure 3.22 depending upon the values of a_1 (which corresponds to ω_n^2, the natural frequency of the system) and a_2.

Don't you just LOVE open-ended problems!?!?!

6) (a) Differential Equations for $\{x_i\}$:

$$\dot{x}_1 = -\alpha x_1 + \beta x_2$$

$$\dot{x}_k = \alpha x_{k-1} - (\alpha + \beta)x_k + \beta x_{k+1}, \quad k = 2, ..., N-1$$

$$\dot{x}_N = \alpha x_{N-1} - \beta x_N$$

(b) $\dot{w} = Aw$, where A is a N X N matrix:

$$\mathbf{A} = \begin{bmatrix} -\alpha & \beta & 0 & .. & 0 \\ \alpha & -\alpha-\beta & \beta & 0 & \vdots \\ 0 & \alpha & -\alpha-\beta & \beta & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \alpha & -\beta \end{bmatrix}$$

(c) Since no liquid leaves the system we have $x_N = -\sum_{i=1}^{N-1} x_i$. Thus with Γ total units of liquid in the system we have

$$\mathbf{A} = \begin{bmatrix} -\alpha & \beta & 0 & .. & 0 \\ \alpha & -\alpha-\beta & \beta & 0 & \vdots \\ 0 & \alpha & -\alpha-\beta & \beta & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ -\beta & -\beta & \cdots & \alpha-\beta & -(\alpha+2\beta) \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \beta\Gamma \end{bmatrix}$$

so that $\dot{\mathbf{v}} = \mathbf{A}v + \mathbf{B}$.

(d) From part(a) we have,$\dot{x}_1 = -x_1$. Solving this differential equation for $x_1(0) = 1$ gives $\dot{x}_1(t) = e^{-t}$. Thus, $\dot{x}_2 + x_2 = \mathbf{e}^{-t}$ so that $x_2(t) = t\mathbf{e}^{-t}$. Likewise $x_3(t) = \frac{t^2}{2}\mathbf{e}^{-t}$ and so on for $x_k(t) = \frac{t^{k-1}}{(k-1)!}\mathbf{e}^{-t}$ $k \leq N-1$. Thus, $x_N(t) = 1 - \sum_{k=1}^{N-1} \frac{t^{k-1}}{(k-1)!}\mathbf{e}^{-t}$.