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Problem Set 4 SOLUTIONS

**S & B:** Ch.[3], prob. 1:
Using the same method as in the problem 2-2, we can find that

$$\phi(t,\tau) = \begin{bmatrix} \frac{t}{\tau} & 0 \\ 0 & \frac{t}{\tau} \end{bmatrix}$$

Using the formula (3.19), we can get the solution

$$x(t) = \begin{bmatrix} \frac{t}{t_0} & 0 \\ 0 & \frac{t}{t_0} \end{bmatrix} x_0 + \int_{t_0}^{t} \begin{bmatrix} \frac{t}{\tau} & 0 \\ 0 & \frac{t}{\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(\tau) d\tau$$

That is

$$x(t) = \begin{bmatrix} \frac{t}{t_0} & 0 \\ 0 & \frac{t}{t_0} \end{bmatrix} x_0 + \int_{t_0}^{t} \begin{bmatrix} \frac{t}{\tau} \\ \frac{t}{\tau} \end{bmatrix} u(\tau) d\tau$$

Using formula (3.20), we can get

$$y(t) = (t \;\; t) x_0/t_0 + (1 \;\; 1) \int_{t_0}^{t} \begin{bmatrix} \frac{t}{\tau} \\ \frac{t}{\tau} \end{bmatrix} u(\tau) d\tau$$

**S & B:** Ch.[3], prob. 2:
From Problem 2-4, we found the transition matrix as

$$\phi(t,\tau) = \frac{1}{3} \begin{bmatrix} 2+e^{3t} & -1+e^{3t} \\ -2+2e^{3t} & 1+2e^{3t} \end{bmatrix}$$

Using formula (3.19), we can get the solution

$$x(t) = \begin{bmatrix} \frac{1}{3}+\frac{2}{3}e^{3t} \\ -\frac{1}{3}+\frac{4}{3}e^{3t} \end{bmatrix} + \int_{0}^{t} \begin{bmatrix} \frac{2}{3}+\frac{1}{3}e^{3(t-\tau)} \\ -\frac{2}{3}+\frac{2}{3}e^{3(t-\tau)} \end{bmatrix} u(\tau) d\tau$$

In case of $u(t)=1$,

$$x(t) = \begin{bmatrix} \frac{2}{9}+\frac{2}{3}t+\frac{7}{9}e^{3t} \\ -\frac{5}{9}-\frac{2}{3}t+\frac{14}{9}e^{3t} \end{bmatrix}$$

Comparing with the result of Problem 2-4, they are the same.

**S & B:** Ch.[3], prob. 6:
Assuming the A, B, C, and D are all constant matrices, taking the Laplace Transfrom of the state equations, we have $sX(s) - x_0 = AX(s) + BU(s)$, and $Y(s) = CX(s) + DU(s)$. We

can solve for $X(s) = (sI - A)^{-1}(BU(s) + x_0)$. We then have $[C(sI-A)^{-1}B + D]U(s) = Y(s)$ so that $H(s) = [C(sI-A)^{-1}B + D]$.

**S & B:** Ch.[3], prob. 7:
Using $e^{At} = T^{-1}e^{\Lambda t}T$, we can get

$$e^{At} = \begin{bmatrix} 1 & \frac{1}{6} - \frac{1}{6}e^{-6t} \\ 0 & e^{-6t} \end{bmatrix}$$

by showing that

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 6 \end{bmatrix} \quad T = \begin{bmatrix} 1 & -\frac{1}{6} \\ 0 & \frac{1}{6} \end{bmatrix}$$

Therefore,

$$\phi(t,\tau) = \begin{bmatrix} 1 & \frac{1}{6} - \frac{1}{6}e^{-6(t-\tau)} \\ 0 & e^{-6(t-\tau)} \end{bmatrix}$$

Applying (3.19) and (3.20), we can obtain the solutions as:

$$x(t) = \begin{bmatrix} 1 & \frac{1}{6} - \frac{1}{6}e^{-6(t-t_0)} \\ 0 & e^{-6(t-t_0)} \end{bmatrix} \mathbf{x}(t_0) + \int_{t_0}^{t} \begin{bmatrix} \frac{1}{3} - \frac{1}{3}e^{-6(t-\tau)} \\ 2e^{-6(t-\tau)} \end{bmatrix} u(\tau)d\tau$$

$$y(t) = \begin{bmatrix} 1 & \frac{1}{6} - \frac{1}{6}e^{-6(t-\tau)} \end{bmatrix} \mathbf{x}(0) + \int_{t_0}^{t} \begin{bmatrix} \frac{1}{3} - \frac{1}{3}e^{-6(t-\tau)} \end{bmatrix} u(\tau)d\tau$$

**S & B:** Ch.[3], prob. 14:
The only time $\dot{x}_1 = 0$ is when $x_1 + x_2 = 0$. Similarly, for $\dot{x}_2 = 0$ we must have $x_1 = x_2$. Thus, $x_1 = x_2 = 0$ at the equilibrium. Concisely: $\bar{\mathbf{x}}^T = [0, 0]$ and $\bar{y} = 0$. We define the linearized variable as $\mathbf{z} = \mathbf{x} - \bar{\mathbf{x}}$ and obtain

$$\dot{z}_1 = f_1(\bar{\mathbf{x}}) + \frac{\partial f_1(\bar{\mathbf{x}})}{\partial x_1} z_1 + \frac{\partial f_1(\bar{\mathbf{x}})}{\partial x_2} z_2$$

and

$$\dot{z}_2 = f_2(\bar{\mathbf{x}}) + \frac{\partial f_2(\bar{\mathbf{x}})}{\partial x_1} z_1 + \frac{\partial f_2(\bar{\mathbf{x}})}{\partial x_2} z_2 - 1$$

where $f_1(x_1, x_2) = (x_1 + x_2)^2$ and $f_2(x_1, x_2) = e^{x_1 - x_2}$. We then have

$$\begin{aligned} \dot{z_1} &= 0 \\ \dot{z_2} &= z_1 - z_2 \end{aligned}$$

Finally, we have the linearized output as w

$$w = \frac{\partial(x_1^2 + 2x_2)}{\partial x_1}\bigg|_{x_1 = x_2 = 0} z_1 + \frac{\partial(x_1^2 + 2x_2)}{\partial x_2}\bigg|_{x_1 = x_2 = 0} z_2 = 2z_2$$

**Problem 6** This is one of the annoying things about statespace form. You can't do things that you think you should be able to do. There are at least two different ways to think about it. The most rigorous is to use linear algebra and consider the resolvent matrix $(sI - A)^{-1}$. First note that the transfer function of a differentiator is $H(s) = s$; i.e., the numerator ($s$) contains a higher order term than the denominator (1). The determinant of

$(sI-A)$ must be of order $n$ where $n$ is the order of matrix $A$. The inverse can be found by composing the "adjugate matrix" for $(sI-A)$. If you don't know what this means, don't worry and skip to the next argument which does not involve linear algebra. Since each term in the adjugate matrix is formed by finding the determinant of the smaller matrix formed by deleting corresponding the row and column of $(sI-A)$, the order of each cofactor can be at most $n-1$ in $s$. Finally the inverse is found by dividing the adjugate matrix by the deteminant (which must be of order $n$ in $s$). So, in total $(sI-A)^{-1}$ cannot have any term whose numerator is of higher order than it's denominator. Which means that $H(s) = CR(s)B + D$ cannot have numerator terms of higher order than the denominator. So differentiation is not possible.

The most straightforward method is to simply consider the solution

$$y(t) = C\phi(t,t_0)x_0 + \int_{t_0}^{t} C\phi(t,\tau)Bu(\tau)d\tau + Du(t)$$

Since $u(\tau)$ appears inside an *integral*, there is no way you can obtain $\dot{u}$ in general for $y(t)$. In fact, were it not for the $Du$ term, you could not even obtain $u$ in the output.

Now all this being said, there are very sneaky ways one might try to make an integrator if you had an infinite state system (very nasty idea). But the basic method is based on Fourier transforms. What you do is use the fact that the integral of a sinusoid differs only in sign and scale from its derivative. Thus, you could in principle apply an infinitely sharp filter at each frequency $\omega$ (infinite state filter), integrate, scale properly and add up all the results from all the (infinite number of) frequencies. Needless to say, this is impractical :)

**Problem 7** Expanding the transfer function, we find that $H(s) = 1 + s + \frac{1}{s(s+1)}$ and we have a term in $s$. This implies a derivative which we know we cannot model using the (finite) state-space representation.

**Problem 8** From the transfer function, we have $Y(s) = (\frac{1}{s(s+1)} + 1)U(s)$. Right away we see that $D = 1$ since that is the only way to get $U(s)$ (and not $\frac{U(s)}{\text{polynomial in } s}$) into $y$. This leaves $C(sI-A)^{-1}B = \frac{1}{s(s+1)}$.

Rewrite as a differential equation, $d^2y/dt^2 + dy/dt = u$. Use I/O form to obtain,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

As a check note that

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \quad C(sI-A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s(s+1)}$$

Also note that this decomposition is not unique. There are infinitely many others (remember similarity transforms).