Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 | Systems Analysis | Fall 1995

Problem Set 3 SOLUTIONS

S & B: Chapter 2, Problem 1:
We apply Lipschitz first. The function $f(x,t) = t\sin x$ is obviously continuous. However,

$$|t\sin(x_1(t)) - t\sin(x_2(t))| = |t||\sin(x_1(t)) - \sin(x_2(t))|$$

Note that $\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2} \le 2\sin\frac{a-b}{2}$. Then since $|\sin x| \le |x|$ [1] we have

$$|t\sin(x_1(t)) - t\sin(x_2(t))| \le |t||x_1(t) - x_2(t)|$$

With the interval restricted to be finite, the Lipschitz condition is satisfied

$$|f(x_1,t) - f(x_2,t)| \le L|x_1(t) - x_2(t)|$$

with $L = \max_{t\in I}|t|$, and a unique solution exists. Note that although the interval must be finite, it can be arbitrarily large.

S & B: Chapter 2, Problem 2:
From the problem, we can see that

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{t}x_1(t) \\ \frac{1}{t}x_2(t) \end{bmatrix}$$

So we have $t\dot{x}_1(t) - x_1(t) = 0$ which is variable-separable, so its solution can be shown to be $x_1(t) = at$. We then have $a = x_1(1)$. Such that, $x_1(t) = tx_1(1)$. Similarly, we have $x_2(t) = tx_2(1)$. Rewrite as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$$

By definition (in S& B), the fundamental/transition matrix is $\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$.

S & B: Chapter 2, Problem 3:
Applying the iteration method of (2.7) we obtain $x_0(t) = 1, x_1(t) = 1 + \int_0^t \tau d\tau = 1 + \frac{t^2}{2}, x_2(t) = 1 + \int_0^t \tau(1 + \frac{\tau^2}{2})d\tau = 1 + t^2 + \frac{t^4}{8}$, and so on. It is easy to show that by using finite induction that $x_k(t)$ is the k^{th} degree Taylor's polynomial of $e^{\frac{t^2}{2}}$. Hence, $x_k(t) \to e^{\frac{t^2}{2}}$ as $t \to \infty$. It is also easy to verify the solution since the differential equation is variable-separable and by proper rearrangement we could integrate both sides.

S & B: Chapter 2, Problem 4:
From Problem 1-16, we have $\phi(t,0) = e^{At}$. Using formula (2.23), we can apply integration directly to the elements of matrix. Therefore,

$$x(t) = \begin{bmatrix} \frac{1}{3} + \frac{2}{3}e^{3t} \\ -\frac{1}{3} + \frac{4}{3}e^{3t} \end{bmatrix} + \begin{bmatrix} \int_0^t(\frac{2}{3} + \frac{1}{3}e^{3(t-\tau)}d\tau \\ \int_0^t(-\frac{2}{3} + \frac{2}{3}e^{3(t-\tau)}d\tau \end{bmatrix} = \begin{bmatrix} \frac{2}{9} + \frac{2}{3}t + \frac{7}{9}e^{3t} \\ -\frac{5}{9} - \frac{2}{3}t + \frac{14}{9}e^{3t} \end{bmatrix}$$

S & B: Chapter 2, Problem 6:
From problem 1-16, we can see that $\phi(t,\tau) = A^{t-\tau} = T^{-1}\Lambda^{t-\tau}T$. Using formula (2.44), we have

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-\tau-1}f(\tau) = \begin{bmatrix} \frac{1}{2} + \frac{5}{6}3^t + \frac{1}{3}0^t \\ -1 + \frac{5}{3}3^t - \frac{1}{3}0^t \end{bmatrix}$$

[1] Can prove this if you like by noting that $d|\sin x|/dx \le 1 = d|x|/dx$ for $x \ge 0$ and that $\sin 0 = 0$. Thus, since $|\sin x|$ grows more slowly than $|x|$ for $x \ge 0$ it must be less than $|x|$ in that region. Do a similar job for $x \le 0$ and you're done.

S & B: Chapter 2, Problem 8:
Using the Laplace transform method (2.36) we obtain

$$X(s) = \begin{bmatrix} \frac{2/9}{s} + \frac{2/3}{s^2} + \frac{7/9}{s-3} \\ \frac{-5/9}{s} - \frac{2/3}{s^2} + \frac{14/9}{s-3} \end{bmatrix}$$

By taking the inverse Laplace transform, we get

$$x(t) = \begin{bmatrix} \frac{2}{9} + \frac{2}{3}t + \frac{7}{9}e^{3t} \\ -\frac{5}{9} - \frac{2}{3}t + \frac{14}{9}e^{3t} \end{bmatrix}$$

S & B: Chapter 2, Problem 14:
Since the intial conditions of this second order equation are zero, by taking z-transform we obtain $z^2X(z) + 3zX(z) + 2X(z) = \frac{z}{z-1}$. Thus,

$$X(z) = \frac{z}{z-1}\frac{1}{(z+2)(z+1)} = \frac{1/6}{z-1} + \frac{1/2}{z+1} + \frac{-2/3}{z+2}$$

by doing the partial fractions expansion. Taking the inverse z-transform, we have $x(t) = \frac{1}{6} + \frac{1}{2}(-1)^{t-1} - \frac{2}{3}(-2)^{t-1}$.

S & B: Chapter 2, Problem 15:
Let: $x_1(t) = x(t)$ and $x_2(t) = x_1(t+1)$. We can get $x_2(t+1) = x(t+2) = 1 - 2x_1(t) - 3x_2(t)$. It can be rewritten in the matrix form as:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ 1 - 2x_1(t) - 3x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

S & B: Chapter 2, Problem 18:
From problem 1-16 we have the transition matrix as

$$\phi(t,\tau) = \frac{1}{3}\begin{bmatrix} 2 + e^{3(t-\tau)} & -1 + e^{3(t-\tau)} \\ -2 + 2e^{3(t-\tau)} & 1 + 2e^{3(t-\tau)} \end{bmatrix}$$

Using equation 2.23 we have

$$x(t) = \frac{1}{3}\begin{bmatrix} 2 + e^{3(t-t_0)} & -1 + e^{3(t-t_0)} \\ -2 + 2e^{3(t-t_0)} & 1 + 2e^{3(t-t_0)} \end{bmatrix}\mathbf{x}(t_0) + \int_{t_0}^{t} \frac{1}{3}\begin{bmatrix} 2 + e^{3(t-\tau)} & -1 + e^{3(t-\tau)} \\ -2 + 2e^{3(t-\tau)} & 1 + 2e^{3(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which simplifies to

$$x(t) = \frac{1}{3}\begin{bmatrix} 2 + e^{3(t-t_0)} & -1 + e^{3(t-t_0)} \\ -2 + 2e^{3(t-t_0)} & 1 + 2e^{3(t-t_0)} \end{bmatrix}\mathbf{x}(t_0) + \frac{1}{3}\begin{bmatrix} 2(t-t_0) + \frac{1}{3}(e^{3(t-t_0)} - 1) \\ -2(t-t_0) + \frac{2}{3}(e^{3(t-t_0)} - 1) \end{bmatrix}$$

Now let $t_0 = 1$ with $\mathbf{x}^T(1) = [1, 1]$ and $t = 0$. We then have

$$x(0) = \frac{1}{3}\begin{bmatrix} 1 + 2e^{-3} \\ -1 + 4e^{-3} \end{bmatrix} + \frac{1}{9}\begin{bmatrix} e^{-3} - 7 \\ 2e^{-3} + 4 \end{bmatrix}$$

which simplifies to

$$x(0) = \frac{1}{9}\begin{bmatrix} 7e^{-3} - 4 \\ 14e^{-3} + 1 \end{bmatrix}$$

S & B: Chapter 2, Problem 19:
We need to show that $|\mathbf{f}(\mathbf{x}_1, t) - \mathbf{f}(\mathbf{x}_2, t)| \leq L|\mathbf{x}_1(t) - \mathbf{x}_2(t)|$ in some region where all the first partials are bounded. We define a matrix $D\mathbf{f}(\mathbf{x}, t)$ with columns $\frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial x_j}$. The mean value theorem (see example 1.6) says that if $\mathbf{f}()$ is continuous in a neighborhood N, then $\exists \chi \in N$ such that

$$\mathbf{f}(\mathbf{x}_1, t) - \mathbf{f}(\mathbf{x}_2, t) = D\mathbf{f}(\chi, t)(\mathbf{x}_1(t) - \mathbf{x}_2(t))$$

We then take the norm of both sides (we assume compatible norms, pp.18)

$$|\mathbf{f}(\mathbf{x}_1, t) - \mathbf{f}(\mathbf{x}_2, t)| \leq |D\mathbf{f}(\chi, t)| \cdot |\mathbf{x}_1(t) - \mathbf{x}_2(t)|$$

We can then bound

$$|D\mathbf{f}(\chi, t)| \leq \max_t |D\mathbf{f}(\chi, t)| = L$$

and we are guaranteed that the maximum exists because the elements of $D\mathbf{f}(\chi, t)$ are all bounded and we assume a finite interval ♣