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330:501 | Systems Analysis | Fall 1995

Problem Set 2 SOLUTIONS

**S & B: Chapter 1, Problem 10:**
First, let's define $\rho(x, y) = ||x - y||$ and $B(x) = Ax + b$ is a mapping. Therefore, $\rho(B(x), B(y)) = ||B(x) - B(y)|| = ||(Ax + b) - (Ay + b)|| = ||A(x - y)|| \leq ||A|| * ||x - y||$. Since, $||A|| < 1$, if we choose $k = ||A||$, then $\rho(B(x), B(y)) \leq k||x - y|| = k\rho(x, y)$. So B(x) is a contrcation mapping. From Thm 1.3, we can see that this equation has a unique solution (fixed point). **QED**.

**S & B: Chapter 1, Problem 11:**
From Thm 1.8, we know that $|\lambda| \leq ||A||$. And $||A||_1 = \max_i \sum_{j=1}^{3} a_{i,j} = 4$, $||A||_\infty = \max_j \sum_{i=1}^{3} a_{i,j} = 4$, $||A||_F = \sqrt{\sum_{i,j} |a_{i,j}|^2} = \sqrt{14}$. Therefore, $|\lambda| \leq \sqrt{14}$.

**S & B: Chapter 1, Problem 12:**
Apply Thm 1.9 to A: Ball $B_1$, $r_1 = \sum_{j=2}^{3} |a_{i,j}| = 3$, Ball $B_2$, $r_2 = \sum_{j=1, j\neq 2}^{3} |a_{i,j}| = 2$, Ball $B_3$, $r_3 = \sum_{j=1}^{2} |a_{i,j}| = 3$. All balls are closed balls with center at 1. Therefore, $D = B_1 \cup B_2 \cup B_3 = [-2, 4]$.

Apply Thm 1.9 to $A^T$: $B_1$, $r_1 = \sum |a'_{i,j}| = 1$, $B_2$, $r_2 = \sum |a'_{i,j}| = 4$, $B_3$, $r_3 = \sum |a'_{i,j}| = 3$. Again all balls are closed ballls with center at 1. Therefore, $D = [-3, 5]$. The first one gives a tighter bound, which is better.

However, you can also take the intersection of the two to obtain $\lambda \in [-2, 4]$ since the eigenvalues of $A^T$ and $A$ are identical (prove it for fun).

**S & B: Chapter 1, Problem 13:**
It is easy to see that the rank of A is $r = 2$ and $m = n = 3$. Specifically we see that the sum of the first two rows yields the third and the first two rows are linarly independent of each other. The dimension of the null space is $n - r = 1$. We can choose any vector which satisties $Ax = 0$, say $x = \alpha[1, 3, -2]^T$ where $\alpha$ is a nonzero constant.

**S & B: Chapter 1, Problem 14:**
We can first find the eigenvalues of A. To solve $|A - \lambda I| = 0$, we have two eigenvalues $\lambda_1 = 0$, and $\lambda_2 = 5$. We can then find the corresponding eigenvectors $x_1 = [2, -1]^T$ and $x_2 = [1, 2]^T$. So:

$$T^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

and

$$T = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

And

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

from which we can verify that $A = T^{-1}\Lambda T$.

**S & B: Chapter 1, Problem 16:**
First find the eigenvalues of A, which are $\lambda_1 = 0$ and $\lambda_2 = 3$. Then find the two corresponding

eigenvectors, $x_1 = [1, -1]^T$ and $x_2 = [1, 2]^T$. Thus,

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

and

$$T = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

We then have

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$e^{\Lambda t} = \begin{bmatrix} 1 & 0 \\ 0 & e^{3t} \end{bmatrix}$$

And since $A = T^{-1} \Lambda T$ we get

$$e^{At} = T^{-1} e^{\Lambda t} T = \frac{1}{3} \begin{bmatrix} 2 + e^{3t} & -1 + e^{3t} \\ -2 + 2e^{3t} & 1 + 2e^{3t} \end{bmatrix}$$

**S & B: Chapter 1, Problem 17:**

Use the result of Example 1.19, to find

$$e^{At} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$