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330:501 Systems Analysis Fall 1995

POP QUIZ SOLUTIONS

These solutions were done very quickly so there might be errors. However, I wanted to get them on the floor for discussion as soon as possible. As bugs arise, I will post a warning to the newsgroup and post the correction. Keep your eyes peeled!

1. Consider the following systems and determine the number of fixed points, the number of limit points and whether the implied mapping is a contraction over the space indicated for time $t \geq t_0$. If there is not enough information given then say so.

   (a) A frog who never moves is placed somewhere on a frog racecourse at time $t = t_0$.
   Infinite number of fixed points but no limit points since there is no region of convergence around each point.

   (b) A marble is tossed with random trajectory into a slightly tilted muffin tin at time $t = t_0$. If you don't know what a muffin tin is, move on and don't worry.
   This is too complex to explain without a picture, so let's just pass.

   (c) A child asleep on a swing (assume seat can hold the child!).
   If the swing arms are rigid then there is one stable equilibrium (limit point) with the swing a the bottom of its travel. This is also a fixed point. There is another fixed point with the swing inverted but this one is not stable.

   (d) The system
   $$\dot{x} = Ax$$
   where A is a square $n \times n$ matrix of zeros.
   Same as frog problem

   (e) Same as above only the matrix $A = I$.
   All eigenvalues have real parts greater than one so any starting state but zero leads to blow up. Thus, one fixed point at $x = 0$ and no limit points.

   (f) Same as above only the matrix $A = -I$.
   Assymptotically stable system. One limit point and fixed point at $x = 0$

   (g) The system
   $$\dot{x} = Ax$$
   where A has all eigenvalues with negative real parts except for one which has value zero; i.e., the matrix A has rank $n - 1$.
   Let z be the eigenvector associated with the zero eigenvalue. We then have $Az = 0$. So if $x_0 = \alpha z$ (α a constant) then the system stays right there... infinite number of fixed points.

The limit point question is a little trickier. It all hinges on how you define a neighborhood of nonzero measure. I think we'll play with this type question more on the exam.

2. **Basic systems problems:**

(a) Find the transition matrix for the system

$$\dot{x} = \begin{bmatrix} -t & 0 \\ 0 & -\cos t \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

A couple of ways to do this. One is to simply solve the system (it's a pair of decoupled first order equations). The other is to use the result from the midterm and find $e^{\int_{t_0}^{t} A(\tau)d\tau}$. This is a little easier and we get,

$$\phi(t, t_0) = \begin{bmatrix} e^{-t^2/2+t_0^2/2} & 0 \\ 0 & e^{\sin t_0 - \sin t} \end{bmatrix}$$

Does this system have a transfer function? NO! (transfer functions as we know them) are defined for LTI systems. This system is not LTI.

(b) Find the transition matrix for the system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Once again a few ways to go. Could solve the system directly by expanding. Could diagonalize. I find it easier to invert $(sI - A)$ and take the inverse Laplace transform.

So,

$$R(s) = (sI - A)^{-1} = \frac{1}{s(s-2)} \begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix}$$

Which implies

$$\phi(t, \tau) = \begin{bmatrix} \frac{1}{2}(1 + e^{2(t-\tau)}) & -\frac{1}{2}(1 - e^{2(t-\tau)}) \\ -\frac{1}{2}(1 - e^{2(t-\tau)}) & \frac{1}{2}(1 + e^{2(t-\tau)}) \end{bmatrix}$$

Does this system have a transfer function? Let $y = x$ and we have $H(s) = (sI = A)^{-1}B = \frac{1}{s-2}[1, 1]^T$

3. **Stability, Observability, Controllability:**

(a) Which of the systems in the previous problem are stable, asymptotically stable, globally asymptotically stable, BIBO stable?

For the time varying system, any $x_1(t) = x_1(t_0)e^{-t^2/2+t_0^2/2}$ which decays toward zero with t. However, we also have $x_2(t) = x_2(t_0)e^{\sin t_0 - \sin t}$ which oscillates. The question is now whether this oscillation satisfies the stability conditions. Pretty clear it does since the oscillation never ventures beyond $x_2(t_0)e^2$.

As for BIBO we must form the $T(t,\tau) = C(t)\phi(t,\tau)B(\tau)$ matrix. Assuming again that $C = I$ we have

$$T(t,\tau) = [e^{-t^2/2+\tau^2/2}, e^{\sin\tau - \sin t}]^T$$

We then have

$$\lim_{t\to\infty} \int_{t_0}^{t} |e^{-t^2/2+\tau^2/2}| d\tau = \text{finite}$$

$$\lim_{t\to\infty} \int_{t_0}^{t} |e^{\sin\tau - \sin t}| = \infty$$

So not BIBO stable.

For the time invariant system, we have two eigenvalues, one with real part > 0 the other identically zero: so unstable. The T matrix will be composed of terms in e^0 and $e^{2(t-\tau)}$ which will blow up with $t \to \infty$.

(b) Which of the systems are controllable?

Need to compute the gramian for the time-varying system. NOTE: MY INSTINCTS IN CLASS WERE WRONG! Controllability (as defined) has LITTLE to do with whether the uncoupled systems are being driven by the same input. As I've threatened both on the PS8 solutions and in the last class, you will see this issue again. In any case

$$W(t_0,t_1) = \int_{t_0}^{t_1} \begin{bmatrix} e^{-t_0^2+\tau^2} & e^{-t_0^2/2+\tau^2/2} e^{\sin\tau - \sin t_0} \\ e^{-t_0^2/2+\tau^2/2} e^{\sin\tau - \sin t_0} & e^{2(\sin\tau - \sin t_0)} \end{bmatrix} d\tau$$

which is hard to evaluate. My GUESS however, is that this baby is usually nonsingular and therefore COMPLETELY CONTROLLABLE!

The LTI system is not controllable since K is obviously singular.

(c) Which of the systems are observable? Assume that $y = [1,0]x$.

The observability gramian for the time-varying system is

$$M(t_0,t_1) = \int_{t_0}^{t_1} \begin{bmatrix} e^{t_0^2-\tau^2} & 0 \\ 0 & 0 \end{bmatrix} d\tau$$

which is obviously singular everywhere. So the time-varying system is NOT observable.

Conversely, the L matrix for the LTI system is full rank so it is completely observable.

4. FINAL QUESTION!

You are given the system

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(a) Assume $x(0) = 0$ and show how to devise a control $u(t)$ (if possible) which makes the output $y(t) = 1 - e^{-t}$ on $t \in [0, 1]$.

This is actually a single variable control problem since the system is decoupled; i.e., $\dot{x}_1 = -x_1 + u(t)$. Since $y(t) = x_1(t)$ we simply need to find a $u(t)$ which makes $x_1(t) = 1 - e^{-t}$. EASY! Just substitute to get $u(t) = 1$.

(b) Now suppose the output is given by $y = [1, 0, 1]x$. Given $u(t)$ and $y(t)$ on $[t_0, t_1]$ can $x(t_0)$ be uniquely determined?

Observability matrix:

$$L = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -4 & 0 \\ 1 & 0 & -16 \end{bmatrix}$$

which has full rank. So yup, $x(t_0)$ can be determined

(c) Given $u(t) = 0$ and output $y(t) = e^{3t} \sin 2t$ on $t \in [0, 1]$, can $x(0)$ be determined? Please think CAREFULLY.

Impossible situation since $y(t)$ should be a homogeneous response (the drive term $u(t)$ is zero), yet it does not have the proper eigenmodes. That is, the eigenvalues of the system matrix are -1 and $\pm 4j$. This trajectory implies eigenvalues of $3 \pm 2j$. So this was a TRICK QUESTION... but I don't ask such things on real exams so don't worry!