Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 Systems Analysis Fall 1996

FINAL EXAMINATION SOLUTIONS

1. *(30 points)* **Fundamental Potpourri**

   (a) *(10 points)* We all know that a linear system obeys the superposition principle. That is, for a linear transformation $T[]$, if $T[x_1] = y_1$ and $T[x_2] = y_2$ then $T[ax_1 + bx_2] = ay_1 + by_2$ where $a$ and $b$ are constants. Prove that $T[0] = 0$ for a linear system.
   $T[0] = T[ax_1 - ax_1] = ay_1 - ay_1 = 0$

   (b) *(10 points)* Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$ with $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ and $u(t)$ a scalar input. Assume that $\mathbf{x}_0 = 0$ at $t = 0$ and that $u(t) = \delta(t)$, the impulse function.

      i. *(5 points)* Find an expression for the resulting impulse response, $\mathbf{h}(t) = \mathbf{y}(t)$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Your final expression must not contain an integral. Show that $\mathbf{y}(t)$ is obtained from a convolution integral,

      $$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(t - \tau) u(\tau) d\tau$$

      Starting state is zero so

      $$\mathbf{y}(t) = \int_0^t \mathbf{C}\phi(t, \tau)B, u(\tau)d\tau = \mathbf{C}\phi(t, 0)\mathbf{B} = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} = \mathbf{h}(t)$$

      and with $\phi(t, \tau) = e^{\mathbf{A}(t-\tau)}$ we have the convolution intergral.

      ii. *(5 points)* What is $\mathbf{H}(s)$, the Laplace Transform of $\mathbf{h}(t)$? Write your result in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $s$.
      The Laplace transform of $e^{\mathbf{A}t}$ is $(s\mathbf{I} - \mathbf{A})^{-1}$ so $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, just as when we took the Laplace transform of the differential equation directly.

   (c) *(10 points)* A certain device with two terminals (like a capacitor, inductor or resistor) has voltage $A$ across its terminals at time $t = 0$. For $t > 0$ the voltage $v(t)$ is given by $v(t) = Ae^{-t}$ with no input applied. An identical device has voltage $B > A$ across its terminals at time $t = 0$ and its terminal voltage subsequently follows $v(t) = Be^{-2t}$, also with no input applied. Can $v(t)$ be the state of the device? Why/why not?
   Setting $Ae^{-t^*} = Be^{-2t^*}$ with $B > A$ we obtain $t^* = \frac{1}{2}\ln(B/A)$ which is greater than zero. Thus the "states" coincide at $t^*$ with exactly the same input afterward but follow different trajectories afterward. This is impossible if $v(t)$ is the state since identical inputs with identical starting states means identical trajectories.
   Therefore, $v(t)$ is not the state.

2. *(50 points)* **Stability**

   (a) *(10 points)* Can the system with characteristic function $s^3 - 5s^2 + 10s + 100$ be asymptotically stable?

   Nope, one of the coefficients is negative.

(b) *(10 points)* Verify that the system with characteristic function $(s+3)^3$ is stable using the Hurwitz criterion.

Hurwitz array:

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

with determinants 3, 8 and 6, so all roots have negative real parts.

(c) *(10 points)* Quantitatively show that for ANY asymptotically stable LTI system with scalar input, for suitably large $t$ the system response to $u(t) = e^{j\omega t}$ is of the form $y_i(t) = \beta_i e^{j\omega t}$ where the $\beta_i$ are complex constants, $\omega$ is a real constant and $y_i$ is the $i^{th}$ output. Show the relationship between the $\beta_i$ and components of the system transfer function matrix $\mathbf{H}(s)$.

Asymptotically stable implies $\phi(t,0) \to 0$ in $t$. Thus

$$y(t) \approx \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}e^{j\omega\tau}d\tau = e^{j\omega t}\mathbf{C}\int_0^t e^{\mathbf{A}\tau}e^{-j\omega\tau}d\tau\mathbf{B} = e^{j\omega t}\beta$$

where $\beta$ is a column vector with elements $\beta_i$. As $t \to \infty$, the integral becomes the laplace transform of $e^{\mathbf{A}t}$ evaluated at $s = j\omega$. Thus $\beta = \mathbf{H}(s = j\omega)$.

(d) *(10 points)* The LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has $\mathbf{A}$ negative definite. Show that $V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}$ is a Lyapunov function for this system.

$V()$, clearly continuous in $\mathbf{x}$, is of the form $w^Tw \geq 0$ and and has its minimum iff $w = 0$. Since $w$ can only be zero if $x$ is zero (since $\mathbf{A}$ is negative definite) we have $V() \geq 0$ with equality iff $\mathbf{x} = \mathbf{0}$. So $V()$ has a unique minimum. Now we check for monotone decrease with time.

$$\dot{V} = \dot{\mathbf{x}}^T\mathbf{A}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{A}^T\mathbf{A}\dot{\mathbf{x}}$$

which reduces to $\dot{V} = 2\mathbf{x}^T\mathbf{A}^T\mathbf{A}\dot{\mathbf{x}} = 2\dot{\mathbf{x}}^T\mathbf{A}\dot{\mathbf{x}} \leq 0$ with equality iff $\dot{\mathbf{x}} = 0$ because $\mathbf{A}$ is negative definite. So $V()$ is a proper Lyapunov function.

(e) *(10 points)* Prove/disprove that an LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$ with $\mathbf{C}$ and $\mathbf{B}$ square, invertible matrices is BIBO stable **iff** the eigenvalues of matrix $\mathbf{A}$ have only negative real parts.

HINT: Matrix and vector norms will help if you're going to be rigorous.

Negative real parts of eigenvalues $\to$ asymptotically stable $\to$ BIBO stable.

Now assume not BIBO stable and calculate $\mathbf{y}(t)$

$$\mathbf{y}(t) = \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Since $B$ is invertible, we can obtain any $\mathbf{B}\mathbf{u}(\tau)$ we like by appropriate choice of $\mathbf{u}(\tau)$. So we can consider $\mathbf{B}\mathbf{u}(t) = \tilde{\mathbf{u}}(t)$ where $\tilde{\mathbf{u}}(t)$ is any vector function. With $\mathbf{C}$ invertible, we rewrite the equation for $\mathbf{y}(t)$ as to obtain the (slightly) transformed system

$$\tilde{\mathbf{y}}(t) = \mathbf{C}^{-1}\mathbf{y}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\tilde{\mathbf{u}}(\tau)d\tau$$

We then invoke our theorem for BIBO stability and see that since not all the eigenvalues of $\mathbf{A}$ have negative real parts we must have

$$\lim_{t\to\infty}\int_0^t |(e^{\mathbf{A}(t-\tau)})_{ij}|d\tau = \infty$$

for some $ij$. So if the system is not asymptotically stable we have

$$\lim_{t\to\infty} ||\mathbf{C}^{-1}\mathbf{y}(t)|| = \infty$$

and since

$$||\mathbf{Q}\mathbf{x}|| \leq ||\mathbf{Q}|| \cdot ||\mathbf{x}||$$

by our matrix norm theory we have

$$\infty = \lim_{t\to\infty} ||\mathbf{C}^{-1}\mathbf{y}(t)|| \leq ||\mathbf{C}^{-1}|| \cdot \lim_{t\to\infty} ||\mathbf{y}(t)||$$

which implies that $\lim_{t\to\infty} ||\mathbf{y}(t)|| = \infty$. So, with $\mathbf{C}$ and $\mathbf{B}$ invertible, asymptotic stability $\leftrightarrow$ BIBO stability. **QED**

3. *(50 points)* **Similarity and Controllability** Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$. $u(t)$ is a scalar input.

   (a) *(10 points)* It is often easier to work with a *similar* diagonal system. Assume $\mathbf{A}$ is diagonalizable and find a **similar** system of the form

   $$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{D}u$$

   where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$. You must specify the relationship between $\mathbf{z}$ and $\mathbf{x}$, and also show that if any element of the column vector $\mathbf{D}$ is zero, the system cannot be completely controllable.
   We have $\mathbf{\Lambda} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$. Letting $\mathbf{z} = \mathbf{T}\mathbf{x}$ we have

   $$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{T}\mathbf{B}u$$

   For controllability we must have $\mathbf{K} = [\mathbf{TB}|\mathbf{\Lambda}\mathbf{TB}|...|\mathbf{\Lambda}^{n-1}\mathbf{TB}]$ of full rank where $\Lambda$ is a diagonal matrix. If any of the elements of the column vector $\mathbf{TB}$ are zero then a row of the $\mathbf{K}$ matrix will be zero, making full rank impossible.

   (b) *(10 points)* Derive an expression for $\mathbf{z}(t)$ for $t \geq 0$ in terms of $\mathbf{\Lambda}$, $\mathbf{D}$, $u(t)$ and the initial state $\mathbf{z}(0) = \mathbf{0}$.

   $$\mathbf{z}(t) = \int_0^t e^{\mathbf{\Lambda}(t-\tau)}\mathbf{TB}u(\tau)d\tau$$

   So,

   $$\mathbf{z}(t) = e^{\mathbf{\Lambda}t}\begin{bmatrix} \int_0^t u(\tau)e^{-\lambda_1\tau}d\tau & 0 & \cdots & \cdots & 0 \\ 0 & \int_0^t u(\tau)e^{-\lambda_2\tau}d\tau & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \int_0^t u(\tau)e^{-\lambda_n\tau}d\tau \end{bmatrix}\mathbf{D}$$

   (c) *(10 points)* Rewrite your expression in terms of $U(\lambda_i)$, $i = 1, 2, ..., N$ where $U(s)$ is the Laplace transform of $u(t)$ and the $\lambda_i$ are the eigenvalues of matrix $\mathbf{A}$.
   HINT: Think of $u(t)$ as time-limited to $(0, t)$.
   If $u(t)$ is time-limited to $(0,t)$ then $\int_0^t u(\tau)e^{-\lambda_i\tau}d\tau = U(\lambda_i)$. Thus we have

$$
\mathbf{z}(t) = \begin{bmatrix} U(\lambda_1) & 0 & \cdots & \cdots & 0 \\ 0 & U(\lambda_2) & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & U(\lambda_n) \end{bmatrix} e^{\mathbf{\Lambda} t}\mathbf{D}
$$

This implies that $z_i(t) = U(\lambda_i)e^{\lambda_i t}d_i$, $i = 1, 2, ..., n$.

(d) *(10 points)* For

$$
\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & \ln 4 \end{bmatrix}
$$

and

$$
\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Find a control input $u(t)$ of the form

$$
u(t) = \begin{cases} \alpha_0 & 0 \le t < \frac{1}{2} \\ \alpha_1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}
$$

which drives the system from $\mathbf{x}(0) = \mathbf{0}$ to $\mathbf{x}(1) = \mathbf{1}$.

We have

$$
\mathbf{x}(1) = \int_0^{\frac{1}{2}} \alpha_0 \begin{bmatrix} 1 \\ e^{(1-\tau)\ln 4} \end{bmatrix} d\tau + \int_{\frac{1}{2}}^{1} \alpha_1 \begin{bmatrix} 1 \\ e^{(1-\tau)\ln 4} \end{bmatrix} d\tau
$$

which reduces to

$$
\mathbf{x}(1) = \begin{bmatrix} \frac{\alpha_0+\alpha_1}{2} \\ \frac{2\alpha_0+\alpha_1}{\ln 4} \end{bmatrix}
$$

So $\alpha_0 + \alpha_1 = 2$ and $2\alpha_0 + \alpha_1 = \ln 4$ or $\alpha_0 = \ln 4 - 2$ and $\alpha_1 = 4 - \ln 4$.

(e) *(10 points)* For the system with initial and terminal states given in part (d). calculate the associated values of $U(\lambda_1)$ and $U(\lambda_2)$ from part (c). Then take the Laplace transform of your result from part (d), calculate $U(0)$ and $U(\ln 2)$ and compare.

From part (c) we obtain $1 = U(0)$ and $1 = U(\ln 4)e^{\ln 4}$ or $U(\ln 4) = 1/4$.

The Laplace transform of a pulse of height $a$ on $(0, 1/2)$ is $a(1-e^{-s/2})/s$. For the shifted pulse on $(1/2, 1)$ we get $ae^{-s/2}(1-e^{-s/2})/s$ so that

$$
U(s) = \frac{1-e^{-s/2}}{s}\left[(\ln 4 - 2) + (4 - \ln 4)e^{-s/2}\right]
$$

We have $\lambda_1 = 0$ and $\lambda_2 = \ln 4$. So for $s = 0$ we have $U(0) = 1$ (remember that $(1-e^{ax})/x$ goes to $-a$ as $x \to 0$). For $s = \ln 4$ we have $U(\ln 4) = \frac{1}{2}(\ln 4)/2 = \frac{1}{4}$ which jibes with the result of part (c).

4. *(50 points)* **The Doomsday Box**

Rutgera Univera, the world famous Rutgers University graduate student, has been given a black box as shown in FIGURE 1. This box contains a mass $m$ connected to the interior walls of the box through springs each with spring constant $k/2$. When the mass is at the midpoint of the box, the force exerted by the symmetric springs is zero. The box sits on a frictionless rail and can only be translated in the $\pm x$ direction.

Rutgera can translate the box in the $x$ direction and can feel the resultant force exerted by the box, $f(t)$. We can assume she knows the value of $m$ and $k$ and the configuration of mass and springs, but cannot directly observe the velocity or acceleration of the mass inside the box.

The mass $m$ is a thermonuclear device which will detonate if it is not brought exactly to rest by time $t = T$. By "rest" it is meant the mass must remain immobile relative the interior of the box after Rutgera is forced to let go at time $t = T$.

We will assume that the box is long enough that the mass can never strike the sides of the box. Assume the force exerted by a spring is of magnitude $kx$ where $x$ is the compression/expansion.

(a) *(25 points)* Rutgera is allowed control $x(t)$ for $0 \leq t \leq T$, but does not know the initial position and velocity of the mass inside the box. Is Rutgera doomed or not? Why/why not? Assume that Rutgera, like all Rutgers University Graduate Students, can be infinitely fast, infinitely strong, and has perfect sensory acuity. That is, she can apply any $x(t)$, and can measure $f(t)$ exactly.

First we need the equations of motion. The displacement of the mass from center is $x(t) - x_m(t)$ and the force exerted on the mass is $k(x(t) - x_m(t))$. Thus, $m\ddot{x}_m = k(x - x_m)$. The force exerted on Rutgera is the same as that exerted by the springs on the box. That is $f(t) = -k(x - x_m)$. We write this in statespace form as

$$\begin{bmatrix} \dot{x}_m \\ \ddot{x}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} x$$

and

$$f(t) = \begin{bmatrix} k & 0 \end{bmatrix} \begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix} - kx(t)$$

Since we know $x(t)$ we can rewrite the output equation as

$$\tilde{f}(t) = f(t) + kx(t) = \begin{bmatrix} k & 0 \end{bmatrix} \begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix}$$

to get our system into the recognizable $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, $\mathbf{y} = \mathbf{C}\mathbf{x}$ form.

In order to know what input to apply, Rutgera first has to determine the state of the system. She can do this if the system is observable. We calculate

$$\mathbf{R}^T = [\mathbf{C}^T | \mathbf{A}^T \mathbf{C}^T] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

which has full rank so the system is observable.

She can control the mass to any desired end state if the system is controllable. We have

$$\mathbf{K} = [\mathbf{B} | \mathbf{A}\mathbf{B}] = \begin{bmatrix} 0 & k/m \\ k/m & 0 \end{bmatrix}$$

which also has full rank. Therefore, Rutgera can determine the initial state of the system and then apply a control which stops the mass from moving at the specified time.

(b) *(25 points)* Now suppose the restriction on time is removed and Rutgera simply wishes to design a feedback control system, using f(t) as the observable which will be used to calculate an appropriate translation so that the mass eventually settles down. Assuming a position set point of $x_0$, please find all the appropriate transfer functions in FIGURE 1 such that internal mass oscillations die down exponentially.

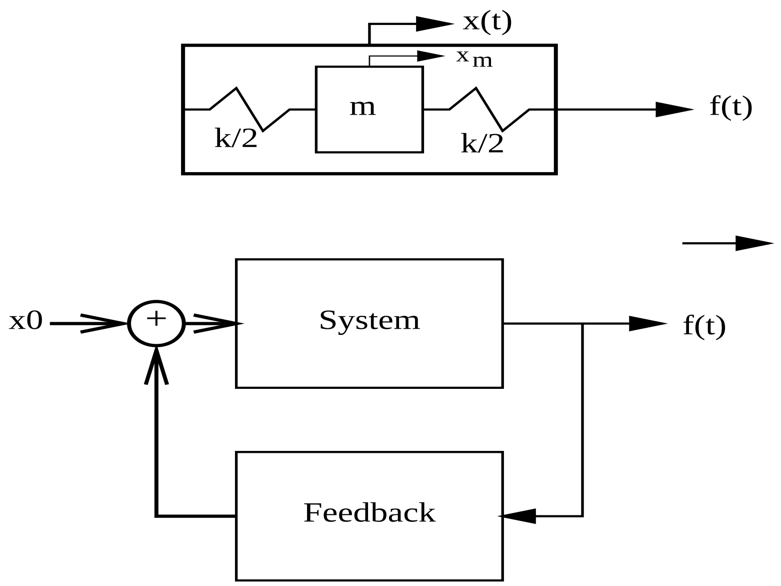


Figure 1: Figure for Doomsday problem

Comment on the relationship between observability, controllability, stability and feedback control in general.

First the comment. What you feed back needs to be related to what you're trying to control. So if you want to control the state, you need to be able to observe the state from the feedback observable and be able to control the state from the feedback control input. If you can do both these things, then you can stabilize the system about any desired operating point.

Now, the transfer function of the system is $\mathbf{H}(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}$. We calculate $(sI - \mathbf{A})^{-1}$ as

$$\frac{1}{s^2 + k/m}\begin{bmatrix} s & 1 \\ -k/m & s \end{bmatrix}$$

Note that we're using the "output" translated system with output $\tilde{f}(t)$. We obtain

$$H(s) = \frac{k}{m}\frac{k}{s^2 + k/m}$$

Using $Q(s)$ as the feedback transfer function we have $(F(s)Q(s) + x_0/s)H(s) = F(s)$ so that the overall transfer function is

$$\frac{F(s)}{x_0/s} = \frac{H(s)}{1 - Q(s)H(s)} = \frac{k^2/m}{s^2 - Q(s)k^2/m + k/m}$$

For stability all we care about is that the denominator of the overal transfer function have roots with real parts less than zero. So let $Q(s) = ams/k^2$. The characteristic equation is then $s^2 - as + k/m$ with roots $\frac{a \pm \sqrt{a^2 - 4k/m}}{2}$. We want all the real portions of the roots to be negative. This will be true for any $a < 0$ – NEGATIVE FEEDBACK STRIKES AGAIN!!!!