Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 Systems Analysis Fall 1996

FINAL EXAMINATION
December 17, 1996

*You have 180 minutes to answer the following questions. The point values are as shown. You are allowed four sides of an 8.5 X 11 sheet of notes for reference. The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT. GOOD LUCK!!!*

1. *(30 points)* **Fundamental Potpourri**

   (a) *(10 points)* We all know that a linear system obeys the superposition principle. That is, for a linear transformation $T[]$, if $T[x_1] = y_1$ and $T[x_2] = y_2$ then $T[ax_1+bx_2] = ay_1+by_2$ where $a$ and $b$ are constants. Prove that $T[0] = 0$ for a linear system.

   (b) *(10 points)* Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$ with $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ and $u(t)$ a scalar input. Assume that $\mathbf{x}_0 = 0$ at $t = 0$ and that $u(t) = \delta(t)$, the impulse function.

      i. *(5 points)* Find an expression for the resulting impulse response, $\mathbf{h}(t) = \mathbf{y}(t)$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Your final expression must not contain an integral. Show that $\mathbf{y}(t)$ is obtained from a convolution integral,

      $$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(t-\tau)u(\tau)d\tau$$

      ii. *(5 points)* What is $\mathbf{H}(s)$, the Laplace Transform of $\mathbf{h}(t)$? Write your result in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $s$.

   (c) *(10 points)* A certain device with two terminals (like a capacitor, inductor or resistor) has voltage $A$ across its terminals at time $t = 0$. For $t > 0$ the voltage $v(t)$ is given by $v(t) = Ae^{-t}$ with no input applied. An identical device has voltage $B > A$ across its terminals at time $t = 0$ and its terminal voltage subsequently follows $v(t) = Be^{-2t}$, also with no input applied. Can $v(t)$ be the state of the device? Why/why not?

2. *(50 points)* **Stability**

   (a) *(10 points)* Can the system with characteristic function $s^3 - 5s^2 + 10s + 100$ be asymptotically stable?

   (b) *(10 points)* Verify that the system with characteristic function $(s+3)^3$ is stable using the Hurwitz criterion.

   (c) *(10 points)* Quantitatively show that for ANY asymptotically stable LTI system with scalar input, for suitably large $t$ the system response to $u(t) = e^{j\omega t}$ is of the form $y_i(t) = \beta_i e^{j\omega t}$ where the $\beta_i$ are complex constants, $\omega$ is a real constant and $y_i$ is the $i^{th}$ output. Show the relationship between the $\beta_i$ and components of the system transfer function matrix $\mathbf{H}(s)$.

(d) *(10 points)* The LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has $\mathbf{A}$ negative definite. Show that $V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}$ is a Lyapunov function for this system.

(e) *(10 points)* Prove/disprove that an LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$ with $\mathbf{C}$ and $\mathbf{B}$ square, invertible matrices is BIBO stable **iff** the eigenvalues of matrix $\mathbf{A}$ have only negative real parts.

HINT: Matrix and vector norms will help if you're going to be rigorous.

3. *(50 points)* **Similarity and Controllability** Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$.

(a) *(10 points)* It is often easier to work with a *similar* diagonal system. Assume $\mathbf{A}$ is diagonalizable and find a **similar** system of the form

$$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{D}u$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$. You must specify the relationship between $\mathbf{z}$ and $\mathbf{x}$, and also show that if any element of $\mathbf{D}$ is zero, the system cannot be completely controllable.

(b) *(10 points)* Derive an expression for $\mathbf{z}(t)$ for $t \geq 0$ in terms of $\mathbf{\Lambda}$, $\mathbf{D}$, $u(t)$ and the initial state $\mathbf{z}(0) = \mathbf{0}$.

(c) *(10 points)* Rewrite your expression in terms of $U(\lambda_i)$, $i = 1, 2, ..., N$ where $U(s)$ is the Laplace transform of $u(t)$ and the $\lambda_i$ are the eigenvalues of matrix $\mathbf{A}$.

HINT: Think of $u(t)$ as time-limited to $(0, t)$.

(d) *(10 points)* For

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & \ln 4 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find a control input $u(t)$ of the form

$$u(t) = \begin{cases} \alpha_0 & 0 \leq t < \frac{1}{2} \\ \alpha_1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

which drives the system from $\mathbf{x}(0) = \mathbf{0}$ to $\mathbf{x}(1) = \mathbf{1}$.

(e) *(10 points)* For the system with initial and terminal states given in part (d), calculate the associated values of $U(\lambda_1)$ and $U(\lambda_2)$ from part (c). Then take the Laplace transform of your result from part (d), calculate $U(\lambda_1)$ and $U(\lambda_2)$ and compare.

4. *(50 points)* **The Doomsday Box**

Rutgera Univera, the world famous Rutgers University graduate student, has been given a black box as shown in FIGURE 1. This box contains a mass $m$ connected to the interior walls of the box through springs each with spring constant $k/2$. When the mass is at the midpoint of the box, the force exerted by the symmetric springs is zero. The box sits on a frictionless rail and can only be translated in the $\pm x$ direction.

Rutgera can translate the box in the $x$ direction and can feel the resultant force exerted by the box, $f(t)$. We can assume she knows the value of $m$ and $k$ and the configuration of mass and springs, but cannot directly observe the velocity or acceleration of the mass inside the box.

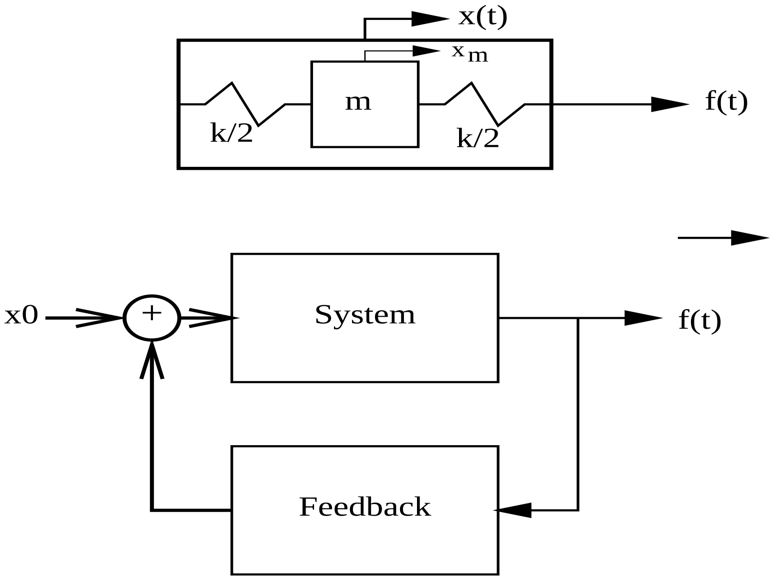


Figure 1: Figure for Doomsday problem

The mass $m$ is a thermonuclear device which will detonate if it is not brought exactly to rest by time $t = T$. By "rest" it is meant the mass must remain immobile relative the interior of the box after Rutgera is forced to let go at time $t = T$.

We will assume that the box is long enough that the mass can never strike the sides of the box. Assume the force exerted by a spring is of magnitude $kx$ where $x$ is the compression/expansion.

(a) *(25 points)* Rutgera is allowed control $x(t)$ for $0 \leq t \leq T$, but does not know the initial position and velocity of the mass inside the box. Is Rutgera doomed or not? Why/why not? Assume that Rutgera, like all Rutgers University Graduate Students, can be infinitely fast, infinitely strong, and has perfect sensory acuity. That is, she can apply any $x(t)$, and can measure $f(t)$ exactly.

(b) *(25 points)* Now suppose the restriction on time is removed and Rutgera simply wishes to design a feedback control system, using f(t) as the observable which will be used to calculate an appropriate translation so that the mass eventually settles down. Assuming a position set point of $x_0$, please find all the appropriate transfer functions in FIGURE 1 such that internal mass oscillations die down exponentially.

Comment on the relationship between observability, controllability, stability and feedback control in general.