Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:501 | Systems Analysis | Fall 1995

FINAL EXAMINATION
December 19, 1995

*You have 180 minutes (if the roof does not collapse from the weight of the snow before then) to answer the following five (5) questions. The point values are as shown. The test is open book: you are allowed*

- *Course textbook*
- *Class notes*
- *Problem sets and solutions*
- *A handwritten course synopsis with relevent formulae*
- *A calculator*

*The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT.*

1. *(20 points)*

   (a) *(5 points)* Is the following a valid transition matrix:

   $$\phi(t,\tau) = \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} + e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix}$$

   Why/Why not?

   (b) *(5 points)* Is the following a valid transition matrix:

   $$\phi(t,\tau) = \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ \sin(t-\tau) & \cos(t-\tau) \end{bmatrix}$$

   Why/Why not?

   (c) *(5 points)* Find the state transition matrix of the system

   $$\dot{x} = \begin{bmatrix} \cos t & 0 \\ 0 & \sin t \end{bmatrix} x$$

   (d) *(5 points)* Find the linear differential equation which corresponds to the state transition matrix:

   $$\phi(t,\tau) = \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix}$$

2. *(20 points)* **Trajectory Control:** Consider the system,

$$\dot{x} = Ax + Bu$$

   (a) *(5 points)* Prove that if $B$ is an $n \times n$ invertible matrix, then for any given state trajectory $x(t)$ there always exists an input $u(t)$ which can drive the state along this trajectory. We will call this property "trajectory controllability".

   (b) *(5 points)* Prove that if an LTI system is *trajectory controllable* as in part (2a), then it is also completely controllable.

   (c) *(5 points)* Prove that if the matrix $B$ has full rank then the system is trajectory controllable. NOTE: This is slightly different than part (2a) since now $B$ is an arbitrary matrix.

   (d) *(5 points)* Prove that if the matrix $B$ does not have rank $n$, then the system is NOT *trajectory controllable*. Thus, complete controllability does not necessarily imply *trajectory controllability*!

3. *(20 points)* **More Trajectory Control!** But you say "if the system is completely controllable, then we should be able to piecewise approximate any given trajectory simply by picking consecutive points along the desired trajectory and driving the system from point to point. Make the pieces close enough together in time and VOILA! we've got the desired trajectory!"

   We will now examine this construction via the following system

$$\dot{x} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

   where $\alpha$ and $\beta$ are real positive constants.

   (a) *(5 points)* Find $W(t_0, t_1)$. Write your result in terms of $\Delta = t_1 - t_0$. Verify that the system is completely controllable.

   (b) *(5 points)* Given two points along the desired trajectory, $x_0$ and $x_1$ (the values at $t_0$ and $t_1$ respectively), find a controlling input $u$ which drives the system from $x_0$ to $x_1$.

   (c) *(5 points)* Your $u(t)$ produces a trajectory which starts at $x_0$ at $t_0$ and ends at $x_1$ at $t_1$. Compute the trajectory $\hat{x}(t)$ produced by your controlling input.

   (d) *(5 points)* The linear approximation to the desired trajectory on $[t_0, t_1]$ is

$$x_\delta(t) = \frac{x_1 - x_0}{\Delta}(t - t_0) + x_0$$

   for small $\Delta$.

   The mean square difference between two functions on interval $[a, b]$ is defined as

$$e^2(f, g) = \frac{1}{b - a} \int_a^b |f(t) - g(t)|^2 dt$$

   Compute the mean square difference between $\hat{x}(t)$ and $x_\delta(t)$. What happens as $\Delta \to 0$? Be careful and specific. Discuss your result in terms of trajectory controllability and complete controllability.

   Extra credit: Your result holds for the input you derived. Does it hold for any possible input which takes the system from $x_0$ to $x_1$ (attempt this AFTER you have finished the examination).

4. *(20 points)* **Fun with Eigenvectors:** You are given the system

$$\dot{x}(t) = Ax(t) + B$$

where $B$ is an $n \times 1$ column vector and all the $n$ eigenvalues of $A$ are distinct.

(a) *(5 points)* $x(0) = 0$ and $B$ is an eigenvector of $A$ with eigenvalue $\lambda$. Compute the trajectory $x(t)$, $t \geq 0$.

(b) *(5 points)* Let $v_i$ be the eigenvectors of $A$ and $\lambda_i$ the associated eigenvalues. Suppose $B = 0$ and $x(0) = \sum_{i=1}^{n} \alpha_i v_i$ where $\{\alpha_i\}$ is a set of constants. Compute the state trajectory $x(t)$, $t \geq 0$.

(c) *(5 points)* Now suppose that $B = \sum_{i=1}^{n} \beta_i v_i$ and $x(0) = 0$ Compute the state trajectory $x(t)$, $t \geq 0$.

(d) *(5 points)* Now suppose that $B = \sum_{i=1}^{n} \beta_i v_i$ and $x(0) = \sum_{i=1}^{n} \alpha_i v_i$. Compute the state trajectory $x(t)$, $t \geq 0$. and relate your result to the standard form

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)Bu(\tau)d\tau$$

5. *(20 points)* **Stability:** You are given a system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x$$

Are the following Lyapunov functions for the system? Why/why not?

(a) *(5 points)*

$$V = (x_1 - x_2)^2 + x_1 x_2$$

(b) *(5 points)*

$$V = (x_1 + x_2)^2$$

(c) *(5 points)*

$$V = 6x_1^2 + x_2^2$$

(d) *(5 points)* Suppose the plant equation is modified to obtain

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 6 & 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Is the system BIBO stable? Why/why not?

HINT: Remember to assume that $x_0 = 0$.