330:501
Solution to Extra Problem Set

1 -

Given

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n,$$

where $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & c \\ c & a & b & 0 & 0 & 0 & 0 \\ 0 & c & a & b & 0 & 0 & 0 \\ 0 & 0 & c & a & b & 0 & 0 \\ 0 & 0 & 0 & c & a & b & 0 \\ 0 & 0 & 0 & 0 & c & a & b \\ b & 0 & 0 & 0 & 0 & c & a \end{bmatrix}$$

with $a = 0.1; b, c = 0.2$. It is obvious that the *one-norm = infinity-norm = 0.5,* and take $\mathbf{z} = \mathbf{x}_{n+1} - \mathbf{x}_n$, then because of compatibility of norms (one or infinity)

$$\|\mathbf{A}\mathbf{z}\| \leq \|\mathbf{A}\| \, \|\mathbf{z}\|$$

so

$$\|\mathbf{A}\mathbf{x}_{n+1} - \mathbf{A}\mathbf{x}_n\| \leq \|\mathbf{A}\| \, \|\mathbf{x}_{n+1} - \mathbf{x}_n\| = 0.5 \, \|\mathbf{x}_{n+1} - \mathbf{x}_n\|$$

thus the mapping is contractive.

2 -

(a)

The matrix difference equation for the demon problem is given by

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$$

where

$$\mathbf{A} = \begin{bmatrix} 2/3 & 1/3 & 0 & 0 & \cdots & 0 \\ 1/3 & 1/3 & 1/3 & 0 & \cdots & 0 \\ 0 & 1/3 & 1/3 & 1/3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & 1/3 & 1/3 & 1/3 \\ 0 & 0 & & 0 & 1/3 & 2/3 \end{bmatrix}.$$

The complete metric space on which the mapping is defined consists of the real non-negative vectors (although it could also be defined on all the real vectors, or even on all the complex vectors for that matter).

First notice that the zero vector is a fixed point of the mapping (even though for this problem we start out with a non-zero vector). We also notice that any vector $[a, a, ...a]$ is also a fixed point. As almost a side issue, for the initial conditions specified in this problem we assume the components must sum to 1 so $a = 1/N$.

So, the mapping cannot be a contraction! Why? Because a contraction mapping can have only one unique fixed point in the complete metrix space over which it is defined. This latter point is why we have to include the zero vector in the space. The space of non-negative vectors without the zero vector is not a complete metric space; i.e., consider the cauchy sequence $[1/n, 1/n, ...1/n]$ which has limit point not in the space.

By the way, if you tried to show that all three of our matrix norms were ≥ 1 that was not enough. Why? Because you did not consider all possible norms. All you need is to find one valid norm (which forms a complete metric space) which is < 1 for the mapping to be a contraction, and it's impossible to consider ALL possible norms. As a concrete example, let's say we only knew about the $|| \cdot ||_\infty$ norm and were given matrix

$$A = \begin{bmatrix} 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 & 0 \end{bmatrix}$$

The column norm is $||A||_\infty = 2$ and we're naively thinking "not a contraction." However, up comes Josephine Smartalek with a definition of some exotic beast called the "row norm" with $||A||_1 = 0.8$ which proves that the mapping using A is a contraction. The fixed point is the zero vector by the way.

This is an important point. We pick our norms for convenience in the problem at hand. Any one will do as long as it is a valid norm (distance measure) and it forms a complete metric space with the region in which we're interested.

(b)

Now the difference equation is given by

$$\mathbf{x}_{n+1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} \mathbf{x}_n,$$

the domain of $\mathbf{A}$ is given by the set of unitary vectors, that is

$$D(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

the range of $\mathbf{A}$ is equal to the domain of $\mathbf{A}$. One can immediately see that there are no fixed points for this mapping, thus it is not a contraction mapping. (Also the one, two and infinity norms of this matrix are equal to one).

3 -

This is interesting because $\mathbf{A}^n$ does not converge to any value as $n \to \infty$, but by simply iterating one can see that $\mathbf{A}^n$ is the nth circular rotation to the left of the rows of $\mathbf{A}$. So that for $n = N$ we obtain an identity matrix.

4 -

For $N = 2$,

$$e^{\mathbf{A}t} = \frac{1}{2} \begin{bmatrix} e^{-t} + e^t & -e^{-t} + e^t \\ -e^{-t} + e^t & e^{-t} + e^t \end{bmatrix}.$$