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College of Engineering
Department of Electrical and Computer Engineering

332:501 — Systems Analysis — Fall 1998

Examination I Statement and Solution

You have THREE WHOLE HOURS to answer the following three (3) questions. The point values are as shown. You are allowed two sides of an 8.5 X 11 sheet of notes for reference. The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT.

GOOD LUCK!

1. *(40 points)* **Contraction Maps and Matrices**

   This problem IS identical to one which appeared on last year's first quiz. Why so easy? Because I want you to CALM DOWN and work through something you know for the first problem. All vectors and matrices are assumed to be elements of $\Re^N$ and $\Re^N \times \Re^N$ respectively.

   (a) *(10 points)* We can state with certainty that the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ is a contraction if:

   i. $||\mathbf{A}||_1 < 1$
   ii. $||\mathbf{A}||_2 < 1$
   iii. $||\mathbf{A}||_\infty < 1$
   iv. (a) or (b) or (c) [logical or]
   v. None of the above

   (**no partial credit on this part**).

   If ANY matrix norm is less than unity, then the mapping is a contraction. The correct answer is (d).

   (b) *(10 points)* Suppose for some $\mathbf{x}^* \neq \underline{0}$ we have $\mathbf{x}^* = \mathbf{A}\mathbf{x}^*$. Can the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ be a contraction over $\Re^N$? Why/why not?

   A contraction mapping has a unique fixed point. Since zero is always a fixed point of this mapping, a nonzero fixed point implies non-uniqueness. Thus the mapping cannot be a contraction. Another way to go was to remember that $|\lambda| \leq ||A||$ WHATEVER norm is used. Since $Ax = x$ implies an eigenvalue of 1, the smallest ANY norm could be is $||A|| = 1$ and that's not small enough for a contraction map.

(c) *(10 points)* We know that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ where $\{\lambda_i\}$, $i = 1, 2, ...N$ are the eigenvalues of matrix $\mathbf{A}$ and $\{\mathbf{v}_i\}$ are the associated eigenvectors. For this part of the problem we assume that the λ_i are distinct so that the $\mathbf{v}_i$ are linearly independent and any vector $\mathbf{x}_0 \in \Re^N$ can be written as

$$\mathbf{x}_0 = \sum_{i=1}^{N} c_i \mathbf{v}_i$$

with a suitable choice of the constants $\{c_i\}$.

Please derive an expression for $\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0$ and **prove** that

$$\lim_{k\to\infty} \mathbf{x}_k = 0$$

for all $\mathbf{x}_0 \in \Re^N$ **iff** $|\lambda_i| < 1$, $i = 1, 2, ...N$.

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 = \mathbf{A}^k \sum_{i=1}^{N} c_i \mathbf{v}_i = \sum_{i=1}^{N} \lambda_i^k c_i \mathbf{v}_i$$

Now

$$0 \leq || \sum_{i=1}^{N} \lambda_i^k c_i \mathbf{v}_i || \leq \sum_{i=1}^{N} |\lambda_i^k| ||c_i \mathbf{v}_i||$$

For arbitrary v_i and c_i, if any of the λ_i have magnitude greater than or equal to 1, the sum cannot go to zero in k. If all the $|\lambda_i| < 1$ then each term goes to zero in k.

(d) *(10 points)* The mapping in the previous part always converged to a fixed point (of zero). Is the condition on the eigenvalues stated in part (1c) necessary and sufficient for the iterative mapping $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ to be a CONTRACTION for all matrices $\mathbf{A}$ with distinct eigenvalues in all metric spaces?

Prove or find and explain a counterexample.

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 100 \\ 0 & \frac{-1}{2} \end{bmatrix}$$

The magnitudes of the two eigenvalues (along the diagonal of this upper-triangular matrix) are both $1/2$. However, both the ρ_1 and ρ_∞ norms are MUCH larger than 1. So $|\lambda_i| < 1$ $\forall i$ is necessary but INSUFFICIENT to guarantee a contraction mapping.

2. *(40 points)* **Grab Bag** (You may NOT use a calculator on this question.)

(a) *(30 points)* An iterative mapping in $\Re^5$ is defined as

$$\mathbf{x}_n = \begin{bmatrix} 0.5 & 0.1 & -0.1 & 0.1 & 0.0 \\ -0.1 & -0.5 & -0.1 & -0.1 & 0.1 \\ -0.1 & -0.1 & 0.6 & -0.0 & 0.1 \\ -0.1 & -0.1 & 0.0 & -0.4 & 0.1 \\ -0.0 & -0.0 & 0.1 & -0.1 & 0.3 \end{bmatrix} \mathbf{x}_{n-1}$$

i. *(10 points)* Does this mapping converge? It if does, find the fixed point. If not, show why not. More rigor = more points.

This is obviously a matrix norm problem with a bit of Gershgorin thrown in to boot. Let's look at the row magnitude sums. For row one it's 0.9. For row two: 0.9, row three: 0.9, row four: 0.7 and row five: 0.5. For later, we also calculate the radii of the closed balls as $r_1 = 0.3$, $r_2 = 0.4$, $r_3 = 0.3$, $r_4 = 0.3$, $r_5 = 0.2$ *and* $r_1 = r_2 = r_3 = r_4 = r_5 = 0.3$ *for the transpose.*

We know that eigenvalues must have magnitude less than the matrix norm. We have both the max row magnitude and column magnitude sum as 0.9 (ρ_∞ *and* ρ_1 *norms). Now let's look at the norm of the vector* $\mathbf{x}_n$*:*

$$||\mathbf{x}_n|| = ||\mathbf{A}\mathbf{x}_{n-1}|| \leq ||\mathbf{A}||||\mathbf{x}_{n-1}|| = 0.9||\mathbf{x}_{n-1}||$$

for either the ρ_∞ *or* ρ_1 *norms. So*

$$||\mathbf{x}_n|| \leq (0.9)^n ||\mathbf{x}_0||$$

which approaches zero as $n \to \infty$*. Thus,* $\mathbf{x}_n$ *approaches the zero vector (by the non-zero argument property of norms).*

ii. *(20 points)* Provide bounds on the geometric convergence rate (or divergence rate if the mapping does not converge) of this mapping. Again, more rigor = more points.

Now Gershgorin pops up. We have overbounded the geometric convergence parameter in the previous problem as 0.9. However, what about an underbound. For this we need to locate the smallest possible magnitude eigenvalue. We'll looking to Gershgorin we see that the eigenvalues reside in closed balls $|\lambda - 0.5| \leq 0.3$, $|\lambda - 0.5| \leq 0.4$, $|\lambda - 0.6| \leq 0.3$, $|\lambda - 0.4| \leq 0.3$, $|\lambda - 0.3| \leq 0.2$, *so that the smallest eigenvalue magnitude is 0.1. Also note that the largest possible eigenvalue magnitude is 0.9, corroborating our norm argument in the previous part. Thus, the smallest geometric parameter is 0.1. That is*

$$(0.1)^n ||\mathbf{x}_0|| \leq ||\mathbf{x}_n|| \leq (0.9)^n ||\mathbf{x}_0||$$

Wasn't that NEATO!?

(b) *(10 points)* The Fourier series for a periodic waveform x(t) with period T is defined as

$$x_k = \frac{1}{T}\int_0^T x(t) e^{-j\frac{2\pi}{T}kt} dt$$

and

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j\frac{2\pi}{T}kt}$$

Does the Fourier series reconstruction satisfy our function distance metric

$$\max_{t\in[0,T]} ||x(t) - \sum_{k=-\infty}^{\infty} x_k e^{j\frac{2\pi}{T}kt}|| = 0?$$

HINT: Let

$$x(t) = \begin{cases} 1 & 0 \leq t < 0.5 \\ 0 & 0.5 \leq t < 1 \end{cases}$$

and assume $x(t)$ is periodic with period $T = 1$. Derive the series coefficients x_k (carefully and correctly!) and then determine whether $\sum_{k=-\infty}^{\infty} x_k e^{j\frac{2\pi}{T}kt} = x(t)$ at $t = 0$.

Never ignore hints! Ok, the series coefficients for the hint function are

$$x_k = \int_0^{\frac{1}{2}} e^{-j2\pi kt} dt = \frac{1}{-j2\pi k} e^{-j2\pi kt}\Big|_0^{\frac{1}{2}} = \frac{-1}{j2\pi k}(e^{-j\pi k} - 1)$$

But

$$(e^{-j\pi k} - 1) = (\cos \pi k - j \sin \pi k - 1)$$

So we have

$$x_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{j\pi k} & k \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\hat{x}(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi k} \sin(2\pi kt)$$

because the cosine terms disappear (x_k is odd in k).

This evaluates to $\hat{x}(t=0) = \frac{1}{2} \neq 1$. So the Fourier series does not converge in the $\max_t |x(t) - \hat{x}(t)|$ sense.

I also asked you (for extra credit) to look a short distance away from $t = 0$ since there's an interesting thing called "The Gibbs Phenomenon". At $t = \epsilon > 0$ we have

$$\hat{x}(\epsilon) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi k} \sin(2\pi k\epsilon)$$

Set

$$\epsilon = \frac{\kappa}{2\pi N}$$

where $\kappa \leq 1$. Obviously ϵ will go to zero as $N \to \infty$. Call this limiting value 0^+. We can bound the sum by using the Taylor (actually, the MacLauren) series:

$$z - z^3/3! \leq \sin z \leq z$$

To make life simple, we set $\kappa = 1$ We then have

$$\sum_{k=1}^{N} \frac{2}{\pi k} \left(\frac{k}{N} - \frac{\left(\frac{k}{N}\right)^3}{3!} \right) \leq \sum_{k=1}^{N} \frac{2}{\pi k} \sin(2\pi k\epsilon) \leq \sum_{k=1}^{N} \frac{2}{\pi k} \frac{k}{N}$$

or

$$\frac{2}{\pi} - \frac{1}{3\pi} \frac{1}{N^3} \sum_{k=1}^{N} k^2 \leq \sum_{k=1}^{N} \frac{2}{\pi k} \sin(2\pi k\epsilon) \leq \frac{2}{\pi}$$

A nice identity (which I did not expect you to know, but it's useful nonetheless) of

$$\sum_{k=1}^{N} k^2 = \frac{1}{6} N(N+1)(2N+1)$$

allows us to rewrite this as

$$\frac{2}{\pi} - \frac{1}{18\pi}\frac{N(N+1)(2N+1)}{N^3} \le \sum_{k=1}^{N} \frac{2}{\pi k}\sin(2\pi k\epsilon) \le \frac{2}{\pi}$$

Taking the limit $N \to \infty$ we obtain

$$\frac{1}{2} + \frac{2}{\pi} - \frac{1}{9\pi} \le \hat{x}(t = 0^+) \le \frac{1}{2} + \frac{2}{\pi}$$

or in numbers

$$1.07 \le \hat{x}(t = 0^+) \le 1.13$$

This overshoot at 0^+ is called Gibb's phenomenon and occurs for any discontinuous function which we represent as a Fourier series.

3. *(40 points)* **Differential Equation and Transition Matrix Stuff**

Let $\phi(t, t_0)$ be the transition matrix for a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

We're going to derive two discretization methods for this differential equation.

(a) *(10 points)* We can approximate $\dot{\mathbf{x}}$ as

$$\dot{\mathbf{x}}(t) \approx \frac{\mathbf{x}(t+\Delta) - \mathbf{x}(t)}{\Delta}$$

for some suitably small Δ. Given $\mathbf{x}(t = 0) = \mathbf{x}_0$, please derive an iterative mapping for $\hat{\mathbf{x}}(n\Delta)$, an approximation to $\mathbf{x}(n\Delta)$.

$$\mathbf{x}(n\Delta) = (\Delta\mathbf{A} + \mathbf{I})\mathbf{x}((n-1)\Delta)$$

so

$$\mathbf{x}(n\Delta) = (\Delta\mathbf{A} + \mathbf{I})^n\mathbf{x}_0$$

(b) *(10 points)* Suppose we are given $\hat{\mathbf{x}}(k\Delta) = \mathbf{q}$. Can we always find the corresponding $\hat{\mathbf{x}}(0)$?. What if anything does our ability to find $\hat{\mathbf{x}}(0)$ have to do with Δ?

Rewrite the mapping as

$$\mathbf{x}(n\Delta) = \Delta^n(\mathbf{A} + \frac{1}{\Delta}\mathbf{I})^n\mathbf{x}_0$$

So we need $(\mathbf{A} + \frac{1}{\Delta}\mathbf{I})$ to be nonsingular if we are to invert the relation and find $\mathbf{x}_0$ from $\mathbf{x}_k$. We have nonsingularity as long as $\frac{-1}{\Delta}$ is not an eigenvalue of $\mathbf{A}$.
It should also be noted that even when the mapping is invertible, it's just an approximation for the real thing.

(c) *(10 points)* Rigorously show that the transtion matrix $\phi(t, t_0) = \phi^{-1}(t_0, t)$.
By definition of the transition matrix, we have

$$x(t) = \phi(t, t_0)x_0$$

and

$$x_0 = \phi(t_0, t)x(t)$$

as unique solutions to initial value problems starting at x_0 *and* $x(t)$ *respectively. Rewriting the first equation we have*

$$\phi^{-1}(t, t_0)x(t) = x_0$$

since the inverse of the transition matrix always exists. However, by the uniqueness property of solutions to the initial value problem, we must have $\phi(t, t_0) = \phi^{-1}(t_0, t)$.

(d) *(10 points)* Please derive an exact (not approximate) iterative mapping for $\mathbf{x}(n\Delta)$ of the form

$$\mathbf{x}(n\Delta) = T[\mathbf{x}((n-1)\Delta)]$$

where $T[]$ is some transformation. Given $\mathbf{x}(n\Delta) = \mathbf{q}$, can we always find $\mathbf{x}(0)$? Why/why not?

$$\mathbf{x}(n\Delta) = \phi^n(\Delta, 0)\mathbf{x}_0$$

and it's always invertible because $\phi(t, t_0)$ *is always invertible. Thus, we can run the equations forward and backward... and they are EXACT.*