Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

332:501 | Systems Analysis | Fall 1998

Examination II Statement and Solutions

You have THREE WHOLE HOURS to answer the following questions. The point values are as shown. You are allowed two sides of an 8.5 X 11 sheet of notes for reference. The problems vary in difficulty so please think through each problem BEFORE you begin to write. DON'T GET STUCK ON ONE PROBLEM. MOVE ON IF YOU ARE STUMPED. YOU MUST SHOW ALL WORK. ANSWERS GIVEN WITHOUT WORK RECEIVE NO CREDIT.

GOOD LUCK!

1. *(40 points)* **Fun With Z-transforms:**

   The final value theorem for Z-Transforms is

   $$\lim_{t\to\infty} x(t) = \lim_{z\to 1}(1-1/z)X(z)$$

   where the Z-transform is defined as

   $$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

   (a) *(10 points)* Show that the theorem is true for $x(t) = 1$, $t \geq 0$.
   $X(z) = \sum_t z^{-t} = 1/(1-1/z)$ *so* $(1-1/z)X(z) = 1$.

   (b) *(10 points)* Evaluate $X(z)$ for $x(t) = (1+(-1)^t)/2$ $(t \geq 0)$ and calculate $\lim_{z\to 1}(1-1/z)X(z)$. Do the same for

   $$x(t) = \begin{cases} 0 & t < 0, t = 1,2,4,5,7,8,... \\ 1 & \text{otherwise} \end{cases}$$

   $X(z) = \sum_{t \; even} z^{-t} = \sum_k z^{-2k} = 1/(1-1/z^2)$ *so* $(1-1/z)X(z) = 1/(1+1/z)$ *which is* $1/2$ *in the limit.*
   Likewise $X(z) = \sum_k z^{-3k} = 1/(1-1/z^3)$ *so* $(1-1/z)X(z) = 1/(1+1/z+1/z^2)$ *which is* $1/3$ *in the limit.*

   (c) *(10 points)* Explain why the theorem does or does not apply to the sequences in the previous part.
   The limit does not exist for those sequences.

(d) *(10 points)* If possible, rigorously derive a "mean value theorem" which states

$$\lim_{z\to 1}(1-1/z)X(z) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} x(k)$$

when $X(z)$ exists.

If not possible, show why the theorem is false.

We artfully rewrite z as $(1+1/N)$ and find that $(1-1/z)X(z)$ evaluated in the limit as $z\to 1$ can be written as

$$\lim_{N\to\infty}\frac{1/N}{1+1/N}\sum_{t=0}^{N-1} x(t)(1+1/N)^{-t}$$

Well,

$$\frac{1/N}{1+1/N}\sum_{t=0}^{N-1} x(t)(1+1/N)^{-t}$$

can be made arbitrarily close to

$$\frac{1}{N}\sum_{t=0}^{N-1} x(t)$$

for N sufficiently large. Take the limit and we have the desired result. We could do sandwiching or epsilon arguments to be even more rigorous if desired.

2. *(40 points)* **Bumpy Roads:**

In class we peeked at the suspension of a vehicle moving along a sinusoidal roadbed. We'll now take a closer look. The system is as shown in FIGURE 1.

A mass (the car body) at position $y(t)$ is suspended by the tires and shock absorbers. These are modeled by a spring with constant K and a dashpot with constant B in parallel connected to the mass of the car body M. The rest length of the suspension (owing to gravity) is L so that when system is at rest and the tire sits at $u(t)=0$, the mass is at $y(t)=L$. Assume the tire is in constant contact with the road.

The roadbed provides an upward displacement $u(t)$ as shown.

(a) *(10 points)* CAREFULLY derive the equations of motion for this system both in standard differential equation form and statespace form. The system input is $u(t)$ and the system output is $y(t)$.

First,

$$-(y-u-L)K-(\dot{y}-\dot{u})B = M\ddot{y}$$

so

$$\ddot{y}+\frac{B}{M}\dot{y}+\frac{K}{M}y = \frac{K}{M}(u+L)+\frac{B}{M}\dot{u}$$

In statespace, letting $\mathbf{x}=[y,\dot{y}]^T$ and $\mathbf{u}=[u(t)+L,\dot{u}]$ we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M}\end{bmatrix}\mathbf{x}+\begin{bmatrix} 0 & 0 \\ \frac{K}{M} & \frac{B}{M}\end{bmatrix}\mathbf{u}$$

with output equation

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Note that we could have also put this directly into input/output form using $\tilde{u} = u + L$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}$$

$$y = \begin{bmatrix} \frac{K}{M} & \frac{B}{M} \end{bmatrix} \mathbf{x}$$

(b) *(10 points)* Derive the transfer function $H(s)$ for this system. Provide a labeled sketch $|H(j\omega)|$ as a function of ω. What is the transfer function value at $\omega = 0$? What is it's value at $\omega = \infty$?

$H(s) = C(sI - A)^{-1}B$. However, it's easier to go directly. from the differential equation. Again let $\tilde{u} = u + L$.

$$H(s) = \frac{Y(s)}{\tilde{U}(s)} = \frac{\frac{K}{M} + \frac{B}{M}s}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

$H(0) = 1$ and $H(\infty) = 0$. Depending upon the values of B, M, and K, it will either generate a peak somewhere around $j\omega = \sqrt{\frac{B^2}{M^2} - 4\frac{K}{M}}$ (if the interior of the radical is negative) or fall off monotonically, at first slowly but then as $1/\omega$. See your textbook for more detail (page 120).

(c) *(10 points)* You are asked to help the world famous Rutgers graduate student, Rutgera Univera to determine parameter values K and B for the suspension. The suspension should be tuned so that

$$|H(j2\pi \times 10\text{Hz})|^2/|H(0)|^2 = \frac{1}{2}$$

Please provide simplified equations from which values of K and B could be obtained and then sketch the magnitude of the resulting transfer function $|H(j\omega)|$. Discuss the physical reasons for your parameter choices.

$H(0) = 1$. Now define $2\xi\omega_0 = \frac{B}{M}$ and $\omega_0^2 = \frac{K}{M}$. Then

$$|H(j\omega)|^2 = \frac{\omega_0^4 + 4\xi^2\omega_0^2\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega_2} = \frac{1}{2}$$

So we need

$$2(\omega_0^4 + 4\xi^2\omega_0^2\omega^2) = (\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega_2$$

or

$$\omega^4 - 2\omega^2\omega_0^2 - \omega_0^4 - 4\xi^2\omega_0^2\omega_2 = 0$$

Rearranging,

$$\omega_0^4 + 2\omega^2\omega_0^2(1 + 2\xi^2) - \omega^4 = 0$$

We can find the appropriate ω_0 in terms of ω and ξ using the quadratic formula. We'll forego that pleasure :).

However, we will look a little at the roots of the characteristic equation,

$$s = -\omega_0\xi \pm \omega_0\sqrt{\xi^2 - 1}$$

If we let $\xi < 1$ then we end up with oscillatory solutions. That's not particularly good. However, if we make ξ huge, then we end up with one eigenvalue very close to zero. That means that the system responds very slowy – not a great thing for a suspension system when you're trying to go over a steep hill.

So what we usually do is "critically damp" the system and set $\xi = 1$. That way we've just killed the oscillations but have not gone overboard and made the system unacceptably slow. Of course, sometimes for fine tuning response speed, you'll just "underdamp" the system and let ξ be close to 1.

(d) *(10 points)* You have now been contacted by Rutgera's evil twin sister, the infamous Aregtur Univera to deploy sinusoidal ($u(t) = \cos \omega t$) speed bumps on the New Jersey Turnpike. Assume your choices for B and K in the previous part, that $\omega = 2\pi v/\lambda$ (where λ is the spacing between crests of the sinusoidal speed bumps) and the subjective judgement that I don't like going up and down faster than about 1/4 time per second when driving at high speed (it scares me). What range of λ values will make a turnpike drive exceptionally annoying for vehicles which try to travel over 100km/hr and whose top speed is 200km/hr. Compare this to your own estimate of the typical speed bump length.

Well, from the previous part you've seen that the system follows the input until about 10Hz and then begins to roll off. If we critically damped the system, then we don't get any annoying resonances near that 10Hz "cutoff" frequency.

So, we want our speed bumps to excite the system where the driver will feel it. That means between 1/4 and 10 Hz. Above 10Hz, the suspension absorbs the vibration and below 1/4Hz the driver doesn't really care. Thus, at 100 km/hr we want the bumps to just begin affecting the driver. This means that $(1/4)Hz = 100km/(3600sec)\lambda$ or $\lambda = 1/9km = 111.11$ meters. That's quite a bit bigger than the usual speed bump. On a school access road where you might want folks to go less than 50 km/hr you'll have a λ of about 55.55 meters.

Of course, for a parking lot where you don't have much space, you're stuck with high narrow ugly speed bumps since the wider ones we've suggested won't fit.

3. *(40 points)* **Swaying Bridges:**

Once when driving into Rutgers from New York City, I got stuck on the George Washington bridge in a traffic jam. Nothing moved for about an hour. During that hour I noticed just how much a bridge deck jostles up and down in the wind. This problem is inspired by that experience.

The abstraction of my situation is depicted in FIGURE 2

(a) *(10 points)* Derive a differential equation which describes the motion along the vertical axis (y axis) of the mass at the center of the bridge. Assume the springs themselves are massless.

If y is the upward displacement of the mass, then each spring stretches by

$$\sqrt{L^2/4 + y^2} - L/2$$

. *This generates a force along the spring of*

$$2K(\sqrt{L^2/4 + y^2} - L/2)$$

but a downward force of only

$$2K(\sqrt{L^2/4+y^2}-L/2)\frac{y}{\sqrt{L^2/4+y^2}}$$

As an aside, notice that we've ignored any initial tension in the spring at $y=0$. This complicates matters only slightly and could be easily included. However, it adds only a bit to the problem. For example, consider what happens in the rest of this problem when we linearize if there is no gravity. I'm not telling since I want you to think about it.

So back to the main issue, the equations of motion are

$$F(t)-Mg-2K(\sqrt{L^2/4+y^2}-L/2)\frac{y}{\sqrt{L^2/4+y^2}}=M\ddot{y}$$

(b) *(10 points)* Linearize this system about some stationary point (not a stationary trajectory) and provide an expression for the natural frequency of oscillation. Assume that an external upward force can be applied to the mass, $F(t)$. With the output assumed to be the vertical displacement from the fixed point, cast the system in statespace form, and in particular, input output form.

Since this is a second order system, you want $\ddot{y}=\dot{y}=0$ for the stationary point. We'll also assume $F(t)=0$.

$$K(\sqrt{L^2/4+y^2}-L/2)\frac{y}{\sqrt{L^2/4+y^2}}=-Mg/2$$

Rather than solve for this value of y, just call it y^*. That's the static "droop" of the center mass. The bigger the mass or the weaker the spring or the lower the initial tension, the more the droop.

Call the linearized variable $\tilde{y}=(y-y^*)$ and we have

$$\ddot{\tilde{y}}=\frac{1}{M}F(t)-2\frac{\partial K(\sqrt{L^2/4+y^2}-L/2)\frac{y}{\sqrt{L^2/4+y^2}}}{\partial y}|_{y=y^*}\tilde{y}$$

or

$$\ddot{\tilde{y}}=\frac{1}{M}F(t)-2K(1-\beta)\tilde{y}$$

where

$$\beta=\frac{(L/2)^3}{(\sqrt{L^2/4+(y^*)^2})^3}$$

The statespace representation is

$$\dot{\tilde{\mathbf{y}}}=\begin{bmatrix}0 & 1\\ -(1-\beta) & 0\end{bmatrix}\tilde{\mathbf{y}}+\begin{bmatrix}1\\0\end{bmatrix}F(t)$$

which is already in I/O form. Please note that $(1-\beta)\geq 0$ so the system has purely imaginary eigenvalues.

(c) *(10 points)* Now allow motion in three dimensions and external forces in three dimensions as well. Provide a differential equation which describes the motion and linearize it about some fixed point (again, not a trajectory).

Well, the easiest way to approach this problem is to use the energy formulation. Let the potential energy of the mass be completely resident in the springs (we'll treat gravity as a constant external force, $F'(t) = F(t) - Mg$) and the kinetic energy as the motion of the mass. We then have

$$\begin{aligned}\mathcal{L} &= \tfrac{M}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &- \tfrac{K}{2}\left[(\sqrt{(x+L/2)^2 + y^2 + z^2} - L/2)^2 + (\sqrt{(x-L/2)^2 + y^2 + z^2} - L/2)^2\right]\end{aligned}$$

from which we form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F_q$$

or

$$\begin{aligned}F_x &= M\ddot{x} \\ &+ K\left[(x+L/2)(1 - \tfrac{L/2}{\sqrt{(x+L/2)^2+y^2+z^2}}) + (x-L/2)(1 - \tfrac{L/2}{\sqrt{(x-L/2)^2+y^2+z^2}})\right] \\ F_y - Mg &= M\ddot{y} \\ &+ K\left[y(1 - \tfrac{L/2}{\sqrt{(x+L/2)^2+y^2+z^2}}) + y(1 - \tfrac{L/2}{\sqrt{(x-L/2)^2+y^2+z^2}})\right] \\ F_z &= M\ddot{z} \\ &+ K\left[z(1 - \tfrac{L/2}{\sqrt{(x+L/2)^2+y^2+z^2}}) + z(1 - \tfrac{L/2}{\sqrt{(x-L/2)^2+y^2+z^2}})\right]\end{aligned}$$

A reasonable stationary point is $x = z = 0$ with $y = y^*$ as before. The linear terms for the x direction:

$$F_x = M\ddot{x} + 2K(1+\alpha)x$$

where $\alpha = \frac{L(y^*)^2/2}{(\sqrt{(L/2)^2+(y^*)^2})^3}$. Notice that the terms in $y - y^*$ and z cancel Likewise for $\tilde{y} = y - y^*$ we have

$$F_y - Mg = M\ddot{\tilde{y}} + 2K(1-\beta)\tilde{y}$$

where as before

$$\beta = \frac{(L/2)^3}{(\sqrt{L^2/4 + (y^*)^2})^3}$$

and for z

$$F_z = M\ddot{z} + 2K(1-\kappa)z$$

where

$$\kappa = \frac{L/2}{\sqrt{(L/2)^2 + (y^*)^2}}$$

.

So we have a set of decoupled second order equations at the fixed point.

(d) *(10 points)* Recast your three-dimensional linearization in input/output form. *Of course, I/O form is for a single input single output system. If you said, I/O form is inappropriate you got full credit. However, you could also just write down THREE independent I/O forms corresponding to the three decoupled equations.*

4. *(40 points)* **Eigenfunctions:**

In the last part of the automobile suspension problem, we considered a specific type of drive function. Such a sinusoidal "speed bump" would be expensive to build since it must cover a large length of the highway (at least as we analyzed it). Suppose now that we want to provide a single speed bump of some given length. How should we design the bump so that it has maximum effect? We will consider this sort of problem for a different system.

(a) *(10 points)* A given system with input $u(t)$ and output $y(t)$ has (acausal) impulse response

$$h(t) = \begin{cases} \sqrt{\frac{3}{2}}(1+t) & -1 \le t \le 0 \\ \sqrt{\frac{3}{2}}(1-t) & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The energy E_j in a signal $f(t)$ is defined as

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Find a simple signal $u(t)$ such that when it is applied to the system, the system output $y(t)$ has unit energy E_y.

An impulse placed into the system will produce $h(t)$. The energy in $h(t)$ is 2/3, so $\sqrt{\frac{3}{2}}\delta(t)$ will do the trick.

(b) *(10 points)* Suppose now that the signal $u(t)$ is constrained to be nonzero only on $(-1, 1)$ and have unit energy. Derive a differential equation for a signal with the property that the output $y(t) = \lambda u(t)$ on $[-1, 1]$: i.e., $u(t)$ is an eigenfunction of the system $h(t)$ with eigenvalue λ on $(-1, 1)$. DO NOT try to solve this differential equation.

If you scale your signal such that it is unit energy ($E_u = 1$), what is the energy of the corresponding $y(t)$ on $[-1, 1]$.

HINT: Try artful differentiation.

First we form the integral equation,

$$\lambda u(t) = \int_{-1}^{1} h(\tau)u(t-\tau)d\tau = \int_{-1}^{0} (1+\tau)u(t-\tau)d\tau + \int_{0}^{1} (1-\tau)u(t-\tau)d\tau$$

Differentiating both sides and playing around a bit (integration by parts and then a final differentiation) yields

$$\frac{d^2 f(t)}{dt^2} = -f(t-1) + 2f(t) - f(t+1)$$

As for the output energy, the output signal will be $\lambda u(t)$ on $[-1, 1]$ by definition, so the energy will be $E_y = \lambda^2$.

(c) *(10 points) This question does not require the answers from parts a) and b).*
Suppose that we can find a set of functions $\{\phi_i(t)\}$ which are all unit energy with eigenvalues $\{\lambda_i\}$ on $t \in [-1, 1]$. Assume that these functions are orthogonal on this interval as well:

$$\int_{-1}^{1} \phi_i(t)\phi_j(t)dt = \delta_{ij}$$

where

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & \text{otherwise} \end{cases}$$

Let $u(t) = \sum_{i=1}^{\infty} a_i\phi_i(t)$ for some given coefficients a_i. What is the energy in $u(t)$ on $[-1, 1]$ in terms of the coefficients $\{a_i\}$? What is the corresponding energy in $y(t)$ on $[-1, 1]$?

The output is $y(t) = \sum_{i=1}^{\infty} a_i\lambda_i\phi_i(t)$ The output energy is $\int_{-1}^{1} y^2(t)dt$. Since all the $\phi_i(t)$ are orthonormal on the interval when we form the product of sums, the cross terms $\phi_i(t)\phi_j(t)$ will integrate to zero unless $i = j$ in which case they'll integrate to 1. Thus,

$$E_y = \sum_{i=1}^{\infty} a_i^2\lambda_i^2$$

This, by the way, is a general form of Parseval's theorem.

(d) *(10 points)* This question does not require the answers from parts a) and b).
Assume that $|\lambda_i| \geq |\lambda_{i+1}|$ $i = 1, 2, \ldots$. Find a unit energy signal $u(t) = \sum_{i=1}^{\infty} a_i\phi_i(t)$ which produces an output $y(t)$ with the maximum possible energy.

What does this have to do with the speed bump problem?

Unit energy input implies $\sum_{i=1}^{\infty} a_i^2 = 1$. Thus, we can rob energy from one component i but must put it back into some combination of the remaining j. Now, notice that if we have two terms of the sum $a_i^2\lambda_i^2$ and $a_j^2\lambda_j^2$ with $|\lambda_j| > |\lambda_i|$ then if we reduce a_i^2 by δ^2 and add δ^2 to a_j^2, then the sum $a_j^2\lambda_j^2 + a_i^2\lambda_i^2$ is increased.
More rigorously:

$$\sum_{i=1}^{\infty} \lambda_i^2 a_i^2 \leq \lambda_1^2 \sum_{i=1}^{\infty} a_i^2 = \lambda_1^2$$

which is achieved with equality when a_i is nonzero only for those i where $\lambda_i = \lambda_1$. Thus, for E_y to be maximum, the only nonzero a_i must be those corresponding to the $\phi_i(t)$ with $\lambda_i = \lambda_1$.

As for the speed bump, one way to think about the problem is that we want to transmit the most energy in the "time" allotted (the interval (-1,1)) to the vehicle. Thus, what we do is make the speed bump profile the same as the maximum eigenvalue eigenfunction of the system response (or a linear combination of all such eigenfunctions with maximum eigenvalue λ^*.

5. *(30 points)* **Mind Expansion:**

(a) *(15 points)* Suppose a class of system has a set of orthogonal (and unit energy) eigenfunctions $\{\phi_i(t)\}$ with corresponding eigenvalues $\{\lambda_i\}$ on some interval $[0, T]$. Further assume that these eigenfunctions are sufficient to represent any time waveform which might be applied to the system.

Describe an appropriate transform pair for use with such a system.

First we project the signal onto the basis functions

$$a_k = \int_0^T x(t)\phi_k(t)dt$$

and to get the signal back, we sum the scaled versions of the basis functions

$$x(t) = \sum_k a_k\phi_k(t)dt$$

(b) *(15 points)* Provide any rigorous example of such a system and transform pair. That is, your example must show that the eigenfunctions are indeed eigenfunctions of the system class you provide and that the eigenfunctions are orthonormal (orthogonal and each with unit energy).

You can forego proving that all functions can be represented by scaled superpositions of your eigenfunctions.

HINT: Don't get fancy.

Any pair will do. Think Fourier.

$$a_k = \int_0^T x(t)e^{-j\frac{2\pi k}{T}t}dt$$

and

$$x(t) = \frac{1}{T}\sum_k a_k e^{j\frac{2\pi k}{T}t}$$

and of course you're wondering about complex conjugation and the scaling right? Well what about orthonormality?

$$\int_0^T e^{j\frac{2\pi k}{T}t}e^{j\frac{2\pi m}{T}t}dt = \int_0^T e^{j\frac{2\pi(k+m)t}{T}}dt = \frac{1}{j\frac{2\pi(k+m)}{T}}e^{j\frac{2\pi(k+m)}{T}t}|_0^T$$

Clearly not orthonormal (aside from the normality part, just set $m = k$ but $m \neq k$ or likewise, $m = -k$). The problem is that for complex functions we need to define the inner product (projection operation) as

$$< f(t), g(t) >= \int_0^T f(t)g^*(t)dt$$

and that fixes the problem (to within a constant of $1/\sqrt{T}$). However, you'd have to be very clever to figure this out by yourself on a test (state it clearly with proper definitions). So another approach is to look at $\sin 2\pi kt/T$ and $\cos 2\pi kt/T$.

$$\int_0^T \sin(\frac{2\pi k}{T}t)\cos(\frac{2\pi m}{T}t)dt = 0$$

and

$$\int_0^T \sin(\frac{2\pi k}{T}t)\sin(\frac{2\pi m}{T}t)dt = \int_0^T \cos(\frac{2\pi k}{T}t)\cos(\frac{2\pi m}{T}t)dt = \frac{T}{2}\delta_{km}$$

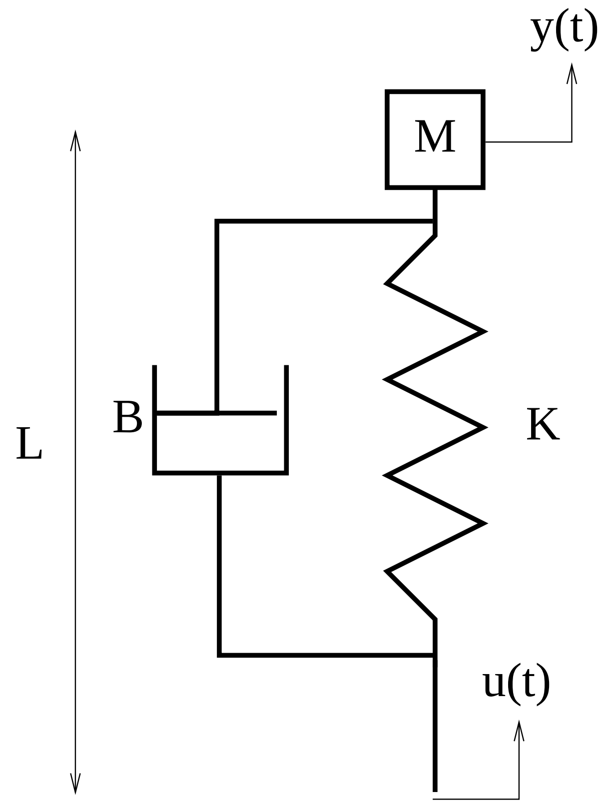

Figure 1: Abstraction of a passive automobile suspension system.

So we then have

$$c_k = \int_0^T x(t)\sqrt{\frac{2}{T}}\cos(\frac{2\pi k}{T}t)dt$$

$$s_k = \int_0^T x(t)\sqrt{\frac{2}{T}}\sin(\frac{2\pi k}{T}t)dt$$

and

$$x(t) = \sum_k s_k\sqrt{\frac{2}{T}}\sin(\frac{2\pi k}{T}t) + \sum_k c_k\sqrt{\frac{2}{T}}\cos(\frac{2\pi k}{T}t)$$

and all is well in a land where cosines and sines are the eigenfunctions – like with linear time-invariant (asymptotically stable) systems.

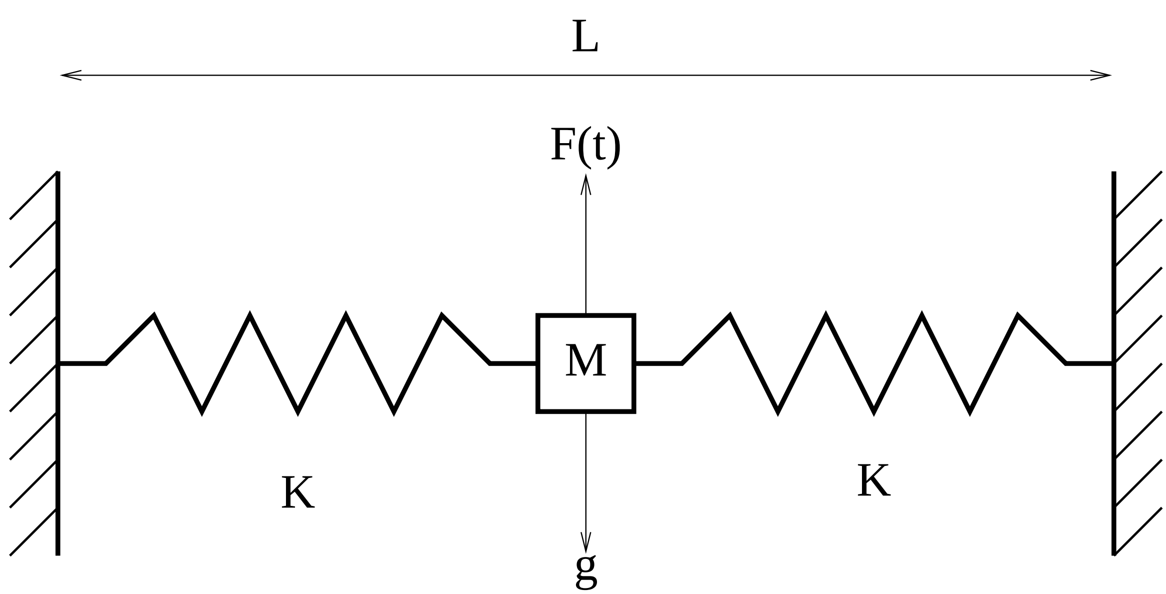

Figure 2: Abstraction of a professor stuck in the middle of a bridge.