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College of Engineering
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332:501 Systems Analysis Fall 1998

Examination III Statement and Solution

1. *(40 points)* **We Begin and End with Basics:**

   Prove (if possible) that if a linear transformation $L()$ is continuous. that $L(x_1+x_2) = L(x_1)+L(x_2)$ implies $L(ax_1+bx_2) = aL(x_1)+bL(x_2)\ \forall a, b \in \Re$. That is, prove that for continuous $L()$, scaling is implied by superposition.

   State your reasons if you think a proof is impossible.

   HINT: Start with integer a and b, proceed to rationals and then irrationals. Be careful at each step.

   We form a sequence of steps:

   $L(2x) = L(x+x) = L(x)+L(x) = 2L(x)$. *Therefore for m a positive integer we have* $L(mx) = L((m-1)x+x) = L((m-1)x)+L(x)$. *Proceeding recursively we obtain* $L(mx) = mL(x)$. *Note also that the same line of reasoning produces* $L(x) = L(0+x) = L(0)+L(x)$ *which implies that* $L(0) = 0$. *We then can say* $L(x-x) = L(x)+L(-x) = L(0) = 0$ *implies that* $L(-x) = -L(x)$. *So now we can form* $L(mx) = mL(x)$ *for all integer m. So superposition implies scaling for integer scale factors.*

   Now as for the rationals, we have $r = m/n$ where m and n are integers. $L(mx/n) = mL(x/n)$. *But we know that* $L(x) = L(n(x/n)) = nL(x/n)$ *so that* $L(x/n) = L(x)/n$ *so that* $L(mx/n) = mL(x)/n$ *and we now have scaling for the rationals.*

   Now what about irrationals? Let I be an irrational number and let r_n be a sequence of rational numbers with $\lim_{n\to\infty} r_n = I$.

   We know $L(r_nx) = r_nL(x)$ *since we've proved it above. Now define* $\epsilon_n = I - r_n$. *We have* $L(Ix) = L(r_nx)+L(\epsilon_nx) = r_nL(x)+L(\epsilon_nx)$. *From our development of metric spaces we know that if $L(x)$ is continuous (or more restrictively, bounded), then if* $\lim_n x_n = x^*$ *we must have* $\lim_n L(x_n) = L(x^*)$. *Since ϵ_n goes to zero in n we have the desired result.*

2. *(40 points)* **The Excitable Capacitor:**

   A certain capacitor has voltage-charge relationship $Q = -CV$ and is arranged in series with a resistor, R, and a voltage source $u(t)$. We will assume the capacitor has zero charge at time $t = 0$ and we desire unit charge at time $t = 1$.

   (a) *(10 points)* Find the differential equation for this series arrangement of resistance R, excitable capacitor, and voltage source $u(t)$. Why is it called an excitable capacitor?

   $$u(t) = i(t)R + V_c(t) = -RC\frac{dV_c(t)}{dt} + V_c(t)$$

It's excitable because if you leave just a little voltage on it, it automatically charges itself without bound. That is, the solution to the homogeneous equation is $Ae^{t/RC}$. So unless you leave this capacitor discharged, or $R = \infty$, you've got a bomb on your hands with time constant $1/RC$.

(b) *(10 points)* Is it always possible to find $u(t)$ which drives the capacitor charge Q to any desired value at any time $t > 0$ given $Q = 0$ at $t = 0$?

NOTE: If you answer simply yes or no, not only will you receive no credit, but in addition, I'll find you next term and drive a stake through your heart.

Statespace:

$$\dot{V}_c = \frac{V_c}{RC} - \frac{u(t)}{RC}$$

Controllability: $K = [-1/RC]$ which is clearly full rank for finite C and R. So in general, we can find a $u(t)$ which drives us from one state point to to any desired state point in some non-zero time. The above problem statement is a special case.

(c) *(10 points)* Find an input $u(t)$ on $(0, 1)$ (any such input, if it exists) which drives the capacitor charge to $Q = 1$ at $t = 1$ assuming $Q = 0$ at $t = 0$.

The simplest (to me) is to set $u(t) = a\delta(t)$ and find the appropriate a. You could also use a step if you're uncomfortable with singularity matching. Impulse in $u(t)$ means corresponding impulse (scaled by $1/RC$) in V_c. This implies a step in V_c at $t = 0^+$: i.e., $V_c(0^+) = \frac{-a}{RC}$. Since $u(t)$ is zero for $t > 0$ we must have

$$V_c(t) = \frac{-a}{RC}e^{\frac{t}{RC}}$$

So to have $Q = 1$ at $t = 1$ we need

$$V_c(1) = -1/C = \frac{-a}{RC}e^{\frac{1}{RC}}$$

which implies $a = Re^{\frac{-1}{RC}}$.

(d) *(10 points)* Now, find an input $u(t)$ on $(0, 1)$ which drives the capacitor charge to $Q = 1$ at $t = 1$ assuming $Q = 0$ at $t = 0$ AND which minimizes the amount of energy E supplied by the source $u(t)$.

HINT: Euler's equation is,

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} - \frac{\partial F}{\partial x} = 0$$

and

$$E = \int_0^1 u(t)i(t)dt$$

where $i(t)$ is the current supplied by the source $u(t)$.

We know $i(t) = -C\dot{V}_c$ and $u(t) = V_c - RC\dot{V}_c$ so

$$E = \int_0^1 -(V_c(t) - RC\dot{V}_c(t))C\dot{V}_c(t)dt$$

We then have $F(V_c, \dot{V}_c, t) = -(V_c - RC\dot{V}_c)C\dot{V}_c$. Expanding

$$-CV_c\dot{V}_c + RC^2(\dot{V}_c)^2$$

taking the appropriate partials and derivatives and setting to zero we obtain

$$-C\dot{V}_c + 2RC^2\ddot{V}_c + C\dot{V}_c = 2RC^2\ddot{V}_c = 0$$

Once again we find that the optimal trajectory for V_c is linear; $V_c(t) = Bt + D$ where B and D are constants. Since we require $V_c(0) = 0$ we have $D = 0$. Since we require $V_c(1) = -q_1/C$ (where $q_1 = 1\,coulomb$) we have $B = -q_1/C$ where B has units of volts per second. Therefore

$$u_o(t) = B(t - RC)$$

3. *(40 points)* **LTI Stability:**

   Determine which (if any) of the following characteristic equations have at least one root with non-negative real part. In each case, state your reasoning (stake through the heart for insufficient answers).

   (a) *(10 points)* $s^3 + s^2 + s - 1$

   Has a negative coefficient so must have at least one root with non-negative real part.

   (b) *(10 points)* $s^3 + s^2 + s + 1$

   All positive coefficients so have to use Hurwitz criterion.

   $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   is not full rank so we must have at least one root with non-negative real part.

   (c) *(10 points)* $s^3 + s^2 + s + 10$

   All positive coefficients so have to use Hurwitz criterion.

   $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   is not full rank so we must have at least one root with non-negative real part.

   (d) *(10 points)* $s^4 + s + 1$ *All positive coefficients so have to use Hurwitz criterion.*

   $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   has negative determinant so there's non-negative real portion of a root lurking in there.

4. *(40 points)* **Lyapunov:**

   (a) *(20 points)* Is the system energy always a Lyapunov function for an isolated non-conservative system? Why/why not? Carefully state you assumptions and remember that a non-conservative system is one with dissipative losses like friction.

Also remember that "isolated" means the system has no external sources of energy.

HINT: First write down the formal conditions a Lyapunov function must satisfy. Assume the strict definition of a Lyapunov function used in your book.

This was (hopefully) a giveaway. The formal statement in your book requires there to be uniquely one point at which the Lyapunov function is minimized. So just imagine a system which consists of a mountain range and a marble. The system is certainly dissipative. But the marble can stop in any of a number of valleys given an appropriate starting point. So, although energy is certainly decreasing, it's not a Lyapunov function in the formal sense. If you balk and say, HEY unless the mountain is ALL THE WAY DOWN the energy is not minimized, just remember that there can be many "lowest" valleys.

(b) *(20 points)* Do non-conservative systems always have a single unique equilibrium? Why/why not? Provide a proof or example.

No. An example was given in the previous part. Another example cited by students was the rolling ball problem from a previous final. Any point on the "flats" is an equilibrium (stable). AND you should also remember that equilibriums need not be stable. Just think of the inververted pendulum (with friction). Two equilibrium points, but only one is stable AND the energy decreases monotonically.

5. *(80 points)* **Rutgera Univera and the Double Pendulum:**

Rutgera Univera has been hired by Positively Pernicious Pendulum, Inc. (PPP) to design a line of mechanical clocks. Needless to say, Rutgera would like to go with the time-tested design of a simple pendulum with small deviation about the vertical. Such a design allows the period of the pendulum to be precisely adjusted by varying the distance of the pendulum bob from the rotation point.

But of course, the PPP executive committee likes the more elaborate 2-bob pendulum design shown in FIGURE 1. They say it has more "pizzazz!" Your job is to determine whether Rutgera can design a stable clock based on the 2-bob pendulum or whether she should immediately quit and find another job since she may be working for technical fools.

(a) *(30 points)* CAREFULLY devise the equations of motion for the 2-bob pendulum. Any errors here will propagate through the problem and you'll lose points on later sections. You may assume completely lossless bearings (no friction) and that the connecting rods are massless and rigid. Furthermore, you should also assume that the bobs are point masses.

HINT: Incorporate gravitational forces through potential energy. Assume the ground position for each mass is it's lowest possible point.

HINT2: Be VERY careful with your algebra and constantly check units to catch errors.

This is an algebraic pain in the butt. However, I figured you should be provided an opportunity to flex your analytic muscle. There's nothing quite like taking a familiar physical system with interesting dynamics and capturing it in differential equations. Better than big game hunting!

So, the potential energy is

$$PE = m_1 g r_1(1-\cos\theta_1) + m_2 g[r_1(1-\cos\theta_1) + r_2(1-\cos\theta_2)]$$

The kinetic energy is

$$KE = \frac{1}{2}m_1(r_1\dot\theta_1)^2 + \frac{1}{2}m_2\left[(r_1\dot\theta_1)^2 + (r_2\dot\theta_2)^2 + 2r_1r_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\right]$$

Letting $\mathcal{L} =$ *KE-PE and taking partials we have*

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -m_2r_1r_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - r_1g(m_1+m_2)\sin\theta_1$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = m_2r_1r_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - m_2r_2g\sin\theta_2$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta_1} = (m_1+m_2)r_1^2\dot\theta_1 + m_2r_1r_2\dot\theta_2\cos(\theta_1-\theta_2)$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta_2} = m_2r_2^2\dot\theta_2 + m_2r_1r_2\dot\theta_1\cos(\theta_1-\theta_2)$$

and now the time derivatives

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta_1} = (m_1+m_2)r_1^2\ddot\theta_1 + m_2r_1r_2\ddot\theta_2\cos(\theta_1-\theta_2) - m_2r_1r_2\dot\theta_2(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta_2} = m_2r_2^2\ddot\theta_2 + m_2r_1r_2\ddot\theta_1\cos(\theta_1-\theta_2) - m_2r_1r_2\dot\theta_1(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2)$$

Now we form the euler equation in each variable.

$$\begin{aligned}
\tfrac{d}{dt}\tfrac{\partial\mathcal{L}}{\partial\dot\theta_1} - \tfrac{\partial\mathcal{L}}{\partial\theta_1} &= (m_1+m_2)r_1^2\ddot\theta_1 + m_2r_1r_2\ddot\theta_2\cos(\theta_1-\theta_2) \\
&- m_2r_1r_2\dot\theta_2(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2) \\
&+ m_2r_1r_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) + r_1g(m_1+m_2)\sin\theta_1 \\
&= (m_1+m_2)r_1^2\ddot\theta_1 + m_2r_1r_2\ddot\theta_2\cos(\theta_1-\theta_2) \\
&+ m_2r_1r_2\dot\theta_2^2\sin(\theta_1-\theta_2) + r_1g(m_1+m_2)\sin\theta_1 \\
&= F_{\theta_1}
\end{aligned}$$

$$\begin{aligned}
\tfrac{d}{dt}\tfrac{\partial\mathcal{L}}{\partial\dot\theta_2} - \tfrac{\partial\mathcal{L}}{\partial\theta_2} &= m_2r_2^2\ddot\theta_2 + m_2r_1r_2\ddot\theta_1\cos(\theta_1-\theta_2) \\
&- m_2r_1r_2\dot\theta_1(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2) \\
&- m_2r_1r_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) + m_2r_2g\sin\theta_2 \\
&= m_2r_2^2\ddot\theta_2 + m_2r_1r_2\ddot\theta_1\cos(\theta_1-\theta_2) \\
&- m_2r_1r_2\dot\theta_1^2\sin(\theta_1-\theta_2) + m_2r_2g\sin\theta_2 \\
&= F_{\theta_2}
\end{aligned}$$

and notice that we have two equations and in each equation both $\ddot\theta_1$ *and* $\ddot\theta_2$ *appear. That does not make us happy campers since we'd really like to move toward state space form. Those of you who stuck with the old rotating book problem noticed a similar conundrum.*

However, we can solve for $\ddot{\theta}_2$ *as*

$$m_2 r_2 \ddot{\theta}_2 = -m_2 r_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 r_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - m_2 g \sin\theta_2 + \frac{1}{r_2} F_{\theta_2}$$

and then substitute to obtain

$$\begin{aligned}&(m_1+m_2) r_1^2 \ddot{\theta}_1 + r_1 \cos(\theta_1 - \theta_2) \left[\begin{array}{c} m_2 r_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \\ -m_2 r_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\ -m_2 g \sin\theta_2 + \frac{1}{r_2} F_{\theta_2} \end{array} \right] \\ &+ m_2 r_1 r_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + r_1 g (m_1 + m_2) \sin\theta_1 \\ &= F_{\theta_1}\end{aligned}$$

which simplifies to yield

$$\begin{array}{c}[m_1 + m_2 \sin^2(\theta_1 - \theta_2)] r_1 \ddot{\theta}_1 + \frac{1}{2} m_2 r_1 \sin[2(\theta_1 - \theta_2)] \dot{\theta}_1^2 \\ + \frac{1}{r_2} \cos(\theta_1 - \theta_2) F_{\theta_2} + m_2 r_2 \sin(\theta_1 - \theta_2) \dot{\theta}_2^2 \\ - m_2 g \sin\theta_2 \cos(\theta_1 - \theta_2) + g(m_1 + m_2) \sin\theta_1 \\ = \frac{1}{r_1} F_{\theta_1}\end{array}$$

Likewise we can isolate $\ddot{\theta}_1$ *as*

$$\ddot{\theta}_1 = \frac{1}{(m_1 + m_2) r_1^2} \left[\begin{array}{c} F_{\theta_1} - m_2 r_1 r_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ -m_2 r_1 r_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - r_1 g (m_1 + m_2) \sin\theta_1 \end{array} \right]$$

Substituting yields

$$\begin{aligned}&m_2 r_2 \ddot{\theta}_2 + \frac{m_2}{m_1 + m_2} \cos(\theta_1 - \theta_2) \left[\begin{array}{c} \frac{1}{r_1} F_{\theta_1} - m_2 r_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ -m_2 r_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - g(m_1 + m_2) \sin\theta_1 \end{array} \right] \\ &- m_2 r_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 \\ &= \frac{1}{r_2} F_{\theta_2}\end{aligned}$$

And simplifying as before we obtain

$$\begin{array}{c}m_2 r_2 (1 - \frac{m_2}{m_1 + m_2} \cos^2(\theta_1 - \theta_2)) \ddot{\theta}_2 - \frac{m_2}{2(m_1 + m_2)} m_2 r_2 \sin[2(\theta_1 - \theta_2)] \dot{\theta}_2^2 \\ + \frac{m_2}{r_1 (m_1 + m_2)} \cos(\theta_1 - \theta_2) F_{\theta_1} - m_2 r_1 \sin(\theta_1 - \theta_2) \dot{\theta}_1^2 \\ + m_2 g [\sin\theta_2 - \sin\theta_1 \cos(\theta_1 - \theta_2)] \\ = \frac{1}{r_2} F_{\theta_2}\end{array}$$

We now have two equations with $\ddot{\theta}_1$ *and* $\ddot{\theta}_2$ *in terms of lower order derivatives of* θ_1 *and* θ_2. *Furthermore, we note (with no small degree of satisfaction) that the multipliers of* $\ddot{\theta}_1$ *and* $\ddot{\theta}_2$ *are always nonzero. Therefore we may completely isolate* $\ddot{\theta}_1$ *and* $\ddot{\theta}_2$ *as*

$$\ddot{\theta}_1 = -\frac{1}{[m_1 + m_2 \sin^2(\theta_1 - \theta_2)] r_1} \left[\begin{array}{c} \frac{1}{2} m_2 r_1 \sin[2(\theta_1 - \theta_2)] \dot{\theta}_1^2 \\ + \frac{1}{r_2} \cos(\theta_1 - \theta_2) F_{\theta_2} + m_2 r_2 \sin(\theta_1 - \theta_2) \dot{\theta}_2^2 \\ - m_2 g \sin\theta_2 \cos(\theta_1 - \theta_2) + g(m_1 + m_2) \sin\theta_1 \\ - \frac{1}{r_1} F_{\theta_1} \end{array} \right]$$

and

$$\ddot{\theta}_2 = -\frac{1}{m_2 r_2(1 - \frac{m_2}{m_1+m_2}\cos^2(\theta_1-\theta_2))}\left[\begin{array}{c} -\frac{m_2}{2(m_1+m_2)} m_2 r_2 \sin[2(\theta_1-\theta_2)]\dot{\theta}_2^2 \\ +\frac{m_2}{r_1(m_1+m_2)}\cos(\theta_1-\theta_2)F_{\theta_1} \\ -m_2 r_1 \sin(\theta_1-\theta_2)\dot{\theta}_1^2 \\ +m_2 g[\sin\theta_2 - \sin\theta_1\cos(\theta_1-\theta_2)] \\ -\frac{1}{r_2}F_{\theta_2} \end{array}\right]$$

We are now in a position to start talking about linearization of the equations and casting them in statespace form.

(b) *(10 points)* Find a stable stationary POINT for this system and linearize about it.

NOTE: Even if you munged the first part, you might still be able to recover here.

The first thing to do is use what approximations we can. First off, the only stable equilibrium is $\theta_1 = \theta_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$. *So we first use small* θ_i *approximations. We will assume that* $m_1/m_2 \gg (\theta_1 - \theta_2)$.

$$\ddot{\theta}_1 \approx -\frac{1}{m_1 r_1}\left[\begin{array}{c} m_2 r_1(\theta_1-\theta_2)\dot{\theta}_1^2 + \frac{1}{r_2}F_{\theta_2} \\ +m_2 r_2(\theta_1-\theta_2)\dot{\theta}_2^2 - m_2 g\theta_2 + g(m_1+m_2)\theta_1 \\ -\frac{1}{r_1}F_{\theta_1} \end{array}\right]$$

and

$$\ddot{\theta}_2 \approx -\frac{1}{m_2 r_2(1 - \frac{m_2}{m_1+m_2})}\left[\begin{array}{c} -\frac{m_2}{(m_1+m_2)} m_2 r_2(\theta_1-\theta_2)\dot{\theta}_2^2 + \frac{m_2}{r_1(m_1+m_2)}F_{\theta_1} \\ -m_2 r_1(\theta_1-\theta_2)\dot{\theta}_1^2 + m_2 g(\theta_2-\theta_1) \\ -\frac{1}{r_2}F_{\theta_2} \end{array}\right]$$

Linearization of the remaining nonlinear terms yields

$$\ddot{\theta}_1 \approx \frac{m_2 g}{m_1 r_1}\theta_2 - \frac{g(m_1+m_2)}{m_1 r_1}\theta_1 + \frac{1}{m_1 r_1^2}F_{\theta_1} - \frac{1}{m_1 r_1 r_2}F_{\theta_2}$$

and

$$\ddot{\theta}_2 \approx \frac{(m_1+m_2)g}{m_1 r_2}(\theta_1-\theta_2) + \frac{m_1+m_2}{m_1 m_2 r_2^2}F_{\theta_2} - \frac{1}{m_1 r_1 r_2}F_{\theta_1}$$

We put it in statespace form for later use. Let $\mathbf{x}^T = [\ \theta_1 \quad \theta_2 \quad \dot{\theta}_1 \quad \dot{\theta}_2\ ]$. *Then we have*

$$\dot{\mathbf{x}} = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g(m_1+m_2)}{m_1 r_1} & \frac{m_2 g}{m_1 r_1} & 0 & 0 \\ \frac{(m_1+m_2)g}{m_1 r_2} & -\frac{(m_1+m_2)g}{m_1 r_2} & 0 & 0 \end{array}\right]\mathbf{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1 r_2^2} & -\frac{1}{m_1 r_1 r_2} \\ -\frac{1}{m_1 r_1 r_2} & \frac{m_1+m_2}{m_1 m_2 r_2^2} \end{array}\right]\left[\begin{array}{c} F_{\theta_1} \\ F_{\theta_2} \end{array}\right]$$

with output equation

$$\theta_1 = y = [\ 1 \quad 0 \quad 0 \quad 0\ ]\mathbf{x}$$

(c) *(10 points)* The output of the system is $\theta_1(t)$, the angular displacement of the upper pendulum rod from the vertical. Derive an expression (if possible) for the period of the linearized system in terms of pendulum bob masses m_1 and m_2 as well as pendulum rod lengths r_1 and r_2.

Well, the characteristic polynomial of the A-matrix is

$$s^4 + (\frac{g(m_1+m_2)}{m_1 r_1} + \frac{(m_1+m_2)g}{m_1 r_2})s^2 + \frac{g^2(m_1+m_2)^2}{m_1^2 r_1 r_2} - \frac{m_2(m_1+m_2)g^2}{m_1^2 r_1 r_2} = 0$$

We see immediately that we'll have roots of the form $s^2 = a, b$ *which means* $s = \pm\sqrt{a}, \pm\sqrt{b}$. *The quadratic formula yields*

$$2s^2 = \frac{g(m_1+m_2)}{m_1}\left[-(\frac{1}{r_1} + \frac{1}{r_2}) \pm \sqrt{\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{2}{r_1 r_2}(1 - 2\frac{m_1}{m_1+m_2})}\right]$$

The term in the radical is at most $(\frac{1}{r_1} + \frac{1}{r_2})^2$ $(m_1 = 0)$ *and at least* $(\frac{1}{r_1} - \frac{1}{r_2})^2$ $(m_1 \to \infty)$. *Therefore* $2s^2 < 0$ *if* $m_1 > 0$ *which means that we have purely imaginary roots*

$$s = \pm\sqrt{\frac{g(m_1+m_2)}{2m_1}}\sqrt{-(\frac{1}{r_1} + \frac{1}{r_2}) \pm \sqrt{\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{2}{r_1 r_2}(1 - 2\frac{m_1}{m_1+m_2})}}$$

That is, there are two natural frequencies. But which is dominant at the pivot point θ_1 *will depend on the relative values of the masses and the relative values of the rod lengths.*

(d) *(30 points)* Assume the input to the double pendulum system is applied through torque (call it $u(t)$) at the upper pivot point which would naturally be attached to the clock housing.

Should Rutgera quit or stay at PPP, Inc.? Be sure to justify your answer by looking at controllability and observability of the linearized system.

This is a tricky question and a number of answers were possible. Since there are two natural frequencies, we could easily have very irregular (aperiodic) motion at the pivot point θ_1. *So, the short answer is that unless the parameters* r_1, r_2, m_1, m_2 *can be precisely chosen so that periodic motion is guaranteed, Rutgera should run away as fast as possible.*

However, I asked you to look at controllability and observability. You probably knew why as you worked the problem. The underlying question is whether it is possible to control the system from the clock housing, maybe using some sort of feedback. I was considering putting this explicitly in the problem, but decided that it would be too much to ask. So, let's just look at controllability first: $K = [B|AB|A^2B|A^3B]$. *However, remember that we're assuming no input* F_{θ_2} *so the second column of the B matrix above disappears and we have*

$$K = \begin{bmatrix} 0 & \frac{1}{m_1 r_2^2} & 0 & -\frac{g(m_1+m_2)}{m_1^2 r_1 r_2^2} - \frac{m_2 g}{m_1^2 r_1^2 r_2} \\ 0 & -\frac{1}{m_1 r_1 r_2} & 0 & \frac{(m_1+m_2)g}{m_1^2 r_2^3} + \frac{(m_1+m_2)g}{m_1^2 r_1 r_2^2} \\ \frac{1}{m_1 r_2^2} & 0 & -\frac{g(m_1+m_2)}{m_1^2 r_1 r_2^2} - \frac{m_2 g}{m_1^2 r_1^2 r_2} & 0 \\ -\frac{1}{m_1 r_1 r_2} & 0 & \frac{(m_1+m_2)g}{m_1^2 r_2^3} + \frac{(m_1+m_2)g}{m_1^2 r_1 r_2^2} & 0 \end{bmatrix}$$

Whether this matrix is full rank boils down to whether the first and third columns are linearly independent and this determination boils down to whether we can find

m_1, m_2, r_1, r_2 such that the determinant of the two by two matrix formed by the lower entries of the first and third columns of K is zero:

$$(\frac{(m_1+m_2)g}{m_1^3 r_2^5} + \frac{(m_1+m_2)g}{m_1^3 r_1 r_2^4}) - (\frac{g(m_1+m_2)}{m_1^3 r_1^2 r_2^3} - \frac{m_2 g}{m_1^3 r_1^3 r_2^2}) = 0$$

which we simplify to

$$(\frac{1}{r_2^3} + \frac{1}{r_1 r_2^2}) + \frac{m_2}{m_1+m_2}\frac{1}{r_1^3} - \frac{1}{r_1^2 r_2} = 0$$

and then put over a common denominator

$$\frac{1}{r_1^3 r_2^3}(r_1^3 + r_1^2 r_2 + \frac{m_2}{m_1+m_2} r_2^3 - r_1 r_2^2) = 0$$

and

$$r_1(r_1^2 - r_2^2) + r_2(r_1^2 + \frac{m_2}{m_1+m_2} r_2^2) = 0$$

So, if $r_1 > r_2$ the expression can never be zero. If $r_1 \leq r_2$ we rearrange

$$(r_1+r_2)r_1^2 + (r_2\frac{m_2}{m_1+m_2} - r_1)r_2^2 = 0$$

Whether this expression is greater than zero depends on the ratio of masses. So the system is not in general controllable. But if Rutgera insists that $r_1 > r_2$, it is controllable. This is a slightly weird condition that I'm not really sure I believe. Check the algebra carefully yourself just in case. I've already been over it a number of times and can find no error, but neither can I establish any physical intuition (for small motions).

If it IS controllable, we could then think about feedback control to stabilize the system (make it really well-behaved). In fact, one could even imagine getting the assembly to behave like a single-bob pendulum ($\theta_1 = \theta_2$). But in order to do that we'd need to know the state of the system so we could tailor our input appropriately.

So it it observable? $M^\top = [C^\top | A^\top C^\top | (A^\top)^2 C^\top | (A^\top)^3 C^\top]$,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{g(m_1+m_2)}{m_1 r_1} & \frac{m_2 g}{m_1 r_1} & 0 & 0 \\ 0 & 0 & -\frac{g(m_1+m_2)}{m_1 r_1} & \frac{m_2 g}{m_1 r_1} \end{bmatrix}$$

which is full rank. So the system is observable.

So maybe Rutgera should stay put and develop really neato clocks with double (or even triple) pendulums. I've never actually seen such a clock, but I think it would be really neat.

Oh and one other tidbit. In general this two pendulum system is chaotic (in the mathematical sense). If you ever are in San Francisco, go to the Exploratorium. Near the entrance they have this two pendulum assembly which you can excite through θ_1 just as we did in this problem. Start gently and it's well behaved. After

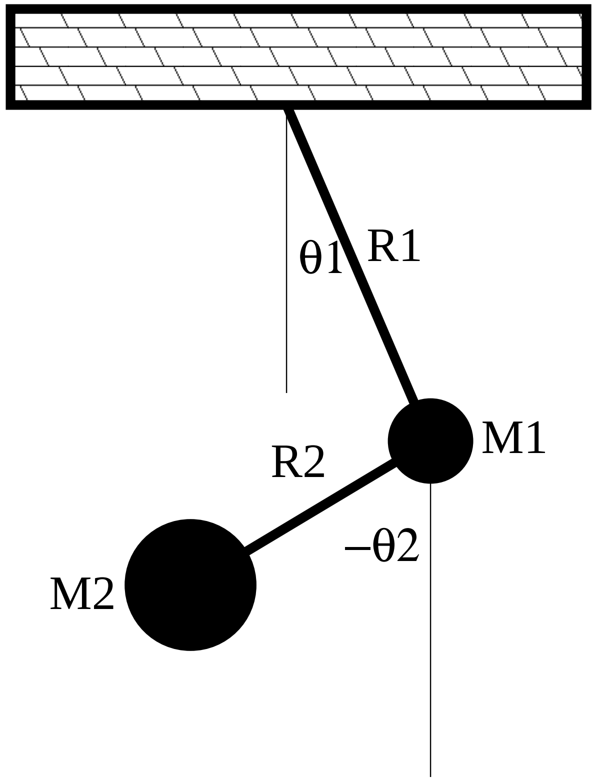

Figure 1: Double Pendulum: all pivot points have full rotational freedom. Input torque may be applied at topmost pivot.

a certain point though, you cannot predict where the thing is going. It jerks around crazily in an engrossing sort of way.

Oh, did I say you CANNOT predict? That's a figure of speech. The system is still observable so in principle, with perfect measurements you could predict the state. The problem is that even small errors in state estimation/observation translate into large errors in where you would have predicted the system to be some time later (chaos).

So, if you ever see someone coming at you with a jointed club, RUN!!!!! Most likely, if you just stay far enough away, the person will club themself to death. But if you stand and fight, you won't be able to predict where the things will strike!

For example, have you ever tried to use nung-chucks (my phonetic spelling)? The number of times I knocked myself in the head (and worse) with the things while trying to be Bruce Lee is emabarassing.