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330:322 Principles of Communication Systems Spring 2004

Math Facts

Leibnitz' Rule

Leibnitz' rule pops up so often in so many things we do that it's worth stating carefully. Suppose you have an integral of the form

$$g(x) = \int_{a(x)}^{b(x)} f(x,t)dt$$

and want to take it's derivative with respect to x. You'd then have

$$\frac{dg(x)}{dx} = \frac{db(x)}{dx} f(x,b(x)) - \frac{da(x)}{dx} f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt$$

Cauchy-Schwarz Inequality

$$\left(\int f(x)g(x)dx \right)^2 \leq \left(\int f^2(x)dx \right) \left(\int g^2(x)dx \right)$$

with equality **iff** $f(x)$ is proportional to $g(x)$.

Convexity

A function $f()$ is convex iff for $\lambda \in [0,1]$ and x_1 and x_2 are any two arguments in the domain of $f()$ we have

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

The function $f()$ is strictly convex if the relation holds with equality iff $\lambda = 0,1$ or $x_1 = x_2$.

The important point about this definition of convexity is that it holds for MULTIVARIATE functions (where the x_i are vectors instead of scalars).

Jensen's Inequality

For a random variable X and a convex function $f()$ we have

$$E[f(X)] \geq f(E[X])$$

If $f()$ is strictly convex, then equality holds only if X is deterministic.

Gaussian Distributions

The general form for a jointly Gaussian distribution on random variables $\{X_i\}$, $i = 1, \ldots, n$ is:

$$f_{\mathbf{X}}(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{n/2} |\mathbf{K}|^{-1/2} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^\top \mathbf{K}^{-1}(\mathbf{x}-\mathbf{m})}$$

where $\mathbf{x}$ is the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and $\mathbf{m} = E[\mathbf{X}]$. The covariance matrix $\mathbf{K}$ is defined as

$$\mathbf{K} = E[\mathbf{X}\mathbf{X}^\top]$$

This general definition allows us to compute the joint Gaussian distribution for two random variables x_1 and x_2 as

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{(x_1-m_1)^2}{\sigma_1^2(1-\rho^2)} - \frac{(x_2-m_2)^2}{\sigma_2^2(1-\rho^2)} + \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2(1-\rho^2)}\right)}$$

where ρ is the correlation coefficient, defined as (usual)

$$\rho = \frac{\mathrm{cov}(x_1,x_2)}{\sigma_1\sigma_2}$$

and $x_i = E[X_i]$.

It is especially important to note that if $\rho = 0$ for *jointly Gaussian* random variables, then they are INDEPENDENT!!! This property holds for more than two jointly Gaussian rv's as well. That is, if the correlation coefficient between any two of the rv's is zero, then they are all mutually independent.

This property does NOT hold in general for other types of random variables.