1. (30 points) Linear Systems:

(a) (10 points) Once again write down the forward and reverse Fourier Transform which relates \(x(t)\) and its Fourier Transform \(X(f)\).

**SOLUTION:**

\[
X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\
x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df
\]

(b) (20 points) Show that if \(x(t)\) has Fourier Transform \(X(f)\), then the Fourier Transform of \(x(t-t_0)\) is \(e^{-j2\pi ft_0}X(f)\).

**SOLUTION:**

\[
\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(s)e^{-j2\pi f(s+t_0)}ds = e^{-j2\pi ft_0}X(f)
\]

2. (30 points) Amplitude Modulation:

(a) (10 points) What is the Fourier Transform of \(m(t)\ cos 2\pi f_c t\ given the Fourier Transform of \(m(t)\ is \(M(f)\)?

**SOLUTION:**

\[
\frac{1}{2}(M(f + f_c) + M(f - f_c))
\]

(b) (10 points) What is the Fourier Transform of \(m(t)\ cos 2\pi f_c t + jm(t)\ sin 2\pi f_c t\ given the Fourier Transform of \(m(t)\ is \(M(f)\)?

**SOLUTION:**

\[
\frac{1}{2}(M(f + f_c) + M(f - f_c)) + j\frac{1}{2j}(-M(f + f_c) + M(f - f_c)) = M(f - f_c)
\]

(c) (10 points) The previous part is an (unrealizable) form of what sort of modulation?

**SOLUTION:** *Single Sideband AM*
3. (30 points) Quantization: (a) (10 points) Why is a quantizer used? State your answer in words (no more than a short paragraph). NOTE: this is not an optimality question, just a simple question about what a quantizer is used for.

**SOLUTION:** The purpose of a quantizer is to approximate samples (usually of a waveform) using a finite set of amplitude levels. Such quantization is a precursor for digital transmission of a signal since samples of a continuous real-valued waveform cannot otherwise be represented with a finite number of bits.

(b) (10 points) The Loyd-Max conditions for optimal quantization are $q_k = E[X|X \in A_k]$ where $A_k$ is the event that random variable $X \in (x_{k-1}, x_k)$ and $x_k = \frac{1}{2}(q_k + q_{k+1})$.

Suppose $f_X(x) = \frac{1}{2}[u(x + 1) - u(x - 1)]$. Is a 1 bit quantizer with $q_0 = -0.5$, $q_1 = 0.5$ and $x_0 = 0$ optimal? Why/Why not?

**SOLUTION:** Yes. $(q_1 + q_0)/2 = 0 = x_0$. $q_0 = E[X|X \in (-1, 0)]$ and $q_1 = E[X|X \in (0, 1)]$.

(c) (10 points) Sketch the output to this quantizer on the interval $t \in (0, 6)$ when the input is the sawtooth waveform $m(t) = u_{-2}(t) + 2 \sum_{k=0}^{\infty} (-1)^k u_{-2}(t - 2k + 1)$

where $u_{-2}(t)$ is the unit ramp (the integral of the unit step). Then provide an analytic expression for $Q(m(t))$ in terms of the unit step function $u(t)$ (also known as $u_{-1}(t)$ in some circles).

**SOLUTION:**

$$Q(m(t)) = \frac{1}{2} \left( u(t) + 2 \sum_{k=1}^{\infty} (-1)^k u(t - 2k + 1) \right)$$