School of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2011**

Quiz I

There are **THREE** problems. Each problem subpart is stated on a different sheet. Show all work on the stapled sheets provided (front and back). DO NOT DETACH THE SHEETS. You are allowed one side of an $8.5 \times 11\text{in}^2$ paper handwritten note sheet.

1. *(50 points)* **Lightning Round (kinda):**

   (a) *(15 points)* What is the Fourier Transform of $x(t) = \cos 2\pi t$? Sketch your result (carefully).
   SOLUTION: *If you got this one wrong, SHAME ON YOU!*
   $X(f) = \frac{1}{2}(\delta(f-1) + \delta(f+1))$

   (b) *(15 points)* Determine whether the following mapping (transformation) $T[m()] = r()$ is linear or nonlinear.

   $$r(t) = \sin\left(2\pi f_c t + \kappa \int_{\infty}^{t} m(x)dx\right)$$

   SOLUTION: *We can do full blown superposition, but let's just trying scaling (homogeneity) first:* $\sin\left(2\pi f_c t + \kappa \int_{\infty}^{t} \alpha m(x)dx\right) \neq \alpha \sin\left(2\pi f_c t + \kappa \int_{\infty}^{t} m(x)dx\right)$ *so definitely NOT LINEAR.*

   (c) *(20 points)* What is the Fourier Transform of $y(t) = \sqrt{\cos 2\pi t}$?
   HINT: This will require a bit more thought and creativity. In fact, at this point **I DO NOT KNOW THE ANSWER!** If an approach does not present itself immediately, I'd suggest skipping this and coming back when you're done with everything else.
   WARNING: And if you tell me you can't take the square root of a negative number, I'll hunt you down and beat you mercilessly with a nerf bat!
   SOLUTION: *We know that*

   $$\mathcal{F}\{\cos 2\pi f_c t\} = \frac{1}{2}(\delta(f-f_c) + \delta(f+f_c))$$

   We also know that

   $$x(t)x(t) \to (X * X)(f)$$

   So if

   $$\mathcal{F}\{\sqrt{\cos 2\pi t}\} = X(f)$$

   we must have

   $$\int_{-\infty}^{\infty} X(\sigma)X(f-\sigma)d\sigma = \frac{1}{2}(\delta(f-1) + \delta(f+1))$$

Now here's where interesting noodling comes in. $\sqrt{\cos 2\pi t}$ is periodic. So we know it's Fourier transform will be composed only of impulses (all the energy is concentrated at multiples of the first harmonic, which in this case is 1Hz). So, we can write $X(f)$ in terms of the Fourier series coefficients of $\sqrt{\cos 2\pi t}$

$$X(f) = \sum_{k=-\infty}^{\infty} a_k \delta(f-k)$$

We then have

$$\int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k \delta(\sigma - k) a_\ell \delta(f-(\sigma-\ell)) d\sigma = \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k a_\ell \delta(f-(k-\ell))$$

or

$$\sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k a_\ell \delta(f-(k-\ell)) = \frac{1}{2}(\delta(f-1)+\delta(f+1))$$

We can rewrite this as

$$\sum_{k=-\infty}^{\infty} a_k a_{k+1} = \frac{1}{2}$$

and

$$\sum_{k=-\infty}^{\infty} a_k a_{k+n} = 0$$

for $n \neq \pm 1$.

And that's basically as far as you can get without using tabulated functions!

Another approach tried by a few students was rather clever I thought – it involved expanding $\sqrt{\cos(2\pi t)}$ in a Taylor series around $t = 0$. No one solved it completely, but you got partial credit for this approach.

2. *(50 points)* **Amplitude Modulation:**

   (a) *(20 points)* Consider an AM signal

   $$r(t) = (\kappa + m(t)) \cos 2\pi f_c t$$

   with program material $m(t)$ bandlimited to $\pm B$ with $B \ll f_c$ and κ chosen so that $\kappa + m(t) \geq 0 \; \forall t$.

   i. *(5 points)* What type of amplitude modulation best describes $r(t)$?
      SOLUTION: *Large Carrier AM*

   ii. *(5 points)* An envelope detector is a rectifier followed by a low pass filter. If $r(t)$ is applied to an ideal envelope detector, what is the output?
      HINT: I'm not asking you to explain the detailed workings of the envelope detector.
      SOLUTION: $\sim (\kappa + m(t))$

   iii. *(10 points)* Suppose the order of the rectifier and the low pass filter were inverted – the low pass applied first followed by rectification. What would the output be?
      SOLUTION: *Zero – the low pass filter completely blocks the incoming high frequency $r(t)$.*

(b) *(30 points)* A synchronous AM frequency-multiplexed system carries baseband spectra $M_n(f)$ each with bandwidth $\pm B$ at frequencies nf_0 so that the total signal on the air is

$$\frac{1}{2}\sum_{n=1}^{N}\left(M(f+nf_0)+M(f-nf_0)\right)$$

i. *(10 points)* What is the maximum allowable value of B to avoid spectrum overlap? Illustate your result with a careful sketch.
SOLUTION: *Carrier separation is f_0 so $B \leq f_0/2$*

ii. *(20 points)* Suppose $N = 100$ and a receiver wishes to recover $M_{50}(f)$ using a heterodyne receiver with bandpass filter bandwidth $\pm W$ for the "sloppy" filter. What is the largest value of W that allows uncorrupted recovery of the desired spectrum if the intermediate frequency is $f_I = 10f_0$? Illustrate your result with a careful sketch.
SOLUTION: *The signal at $50f_0$ moves down to $10f_0$ – a shift of $40f_0$. The original lower band edge (right hand spectrum) is at $f_0 - B$, so the shifted band edge is at $39f_0 - B \rightarrow$ which means the spectrum centered around $-10f_0$ is being clobbered. So, you gotta filter the original below $50f_0 - (20f_0 - B)$ and above $50f_0 + (20f_0 + B)$.*

$$W \leq 20f_0 - B$$

3. *(50 points)* **Cora's Squirrel Moodulator**

Cora the Communications Engineer has once again captured her arch nemesis, Martin T. Sciuridae. She has Marty the squirrel locked in the squirrel equivalent of a straight jacket. Her apparatus applies a feather to Marty's feet which causes Marty to sing a continuous constant amplitude pure tone at frequency $p\Delta f + f_m$ where p is the pressure applied to Marty in a special unit of force Cora calls "Mnads."

(a) *(20 points)* Please provide a plausible expression for $r(t)$ the signal emitted by Marty for $t \geq 0$ if Cora applies pressure $p(t)$ for $t \geq 0$. What is the best description for $r(t)$ in terms of modulation schemes we've covered? State all your assumptions, if any.
SOLUTION: *FM:*

$$r(t) = \cos\left(2\pi f_m t + \int_0^t \Delta f p(x)dx\right)$$

(b) *(10 points)* What type of modulation does $r(t)$ represent if $|p(t)\Delta f| < 0.01$ for $t \geq 0$?
SOLUTION: *Narrowband FM*

(c) *(20 points)* Squirrels are lightweight creatures and can be moved rather rapidly with relatively little energy. So, suppose that now instead of applying Mnads, Cora attaches Marty to a vertical rigid board which she moves back and forth so that the distance between Marty's mouth and a fixed microphone (receiver) varies as a function of time $z(t)$. Assume Marty emits a tone at f_m and that the motion is slow relative this frequency (so you can ignore doppler effects – like how my voice pitch would change as I run screaming at you with the nerf bat if you muff up (1a) on this quiz!). Please provide a plausible expression for the waveform received by the microphone and provide the best description of it in terms of modulation schemes we have covered. Assume the speed of sound in air is v_s. State all other assumptions.

SOLUTION: *PM:*

$$r(t) = \cos\left(2\pi f_m(t - \frac{z(t)}{v_s})\right) = \cos\left(2\pi f_m t - \frac{2\pi f_m z(t)}{v_s})\right)$$