School of Engineering
Department of Electrical and Computer Engineering

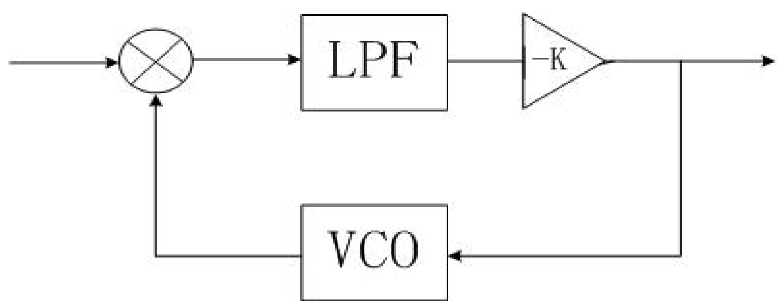
332:322 **Principles of Communications Systems** **Spring 2009**

Quiz II

There are **THREE** problems. Each problem subpart is stated on a different sheet. Show all work on the stapled sheets provided (front and back). DO NOT DETACH THE SHEETS. You are allowed TWO sides of an $8.5 \times 11\text{in}^2$ paper handwritten note sheet.

1. **Phase Locked Loop**

   (a) *(20 points)* Please draw a labeled block diagram of a phase locked loop.
   SOLUTION:



   (b) *(20 points)* Low frequency program material $s(t)$ is used to amplitude-modulate a high frequency sinusoid $\cos(2\pi f_c t)$ and produce $r(t) = s(t)\cos(2\pi f_c t)$. Then, $r(t)$ is applied to the input of a phase locked loop. Describe in words the output of the VCO and the output of the phase locked loop.
   Remember: if the input to a VCO is $\frac{dx(t)}{dt}$ then the output is $\sin x(t)$.
   SOLUTION: *The output of VCO is* $\sin 2\pi f_c t$ *and the output of PLL is* $2\pi f_c$

   (c) *(20 points)* Please justify your previous assertions analytically.
   SOLUTION:
   The input to the multiplier are $s(t)\cos(2\pi f_c t)$ *and* $\sin x(t)$. *We can rewrite* $x(t) = 2\pi f_c t + e(t)$ *where* $e(t)$ *is assumed small.*
   Output of multiplier:

$$
\begin{aligned}
&= s(t)\cos(2\pi f_c t)\sin x(t) \\
&= s(t)\cos(2\pi f_c t)[\sin(2\pi f_c t)\cos e(t) + \cos(2\pi f_c t) sine(t)] \\
&= s(t)\sin(4\pi f_c t) + \tfrac{1}{2}\cos(4\pi f_c t)e(t) + \tfrac{1}{2}e(t)
\end{aligned}
$$

   Only the third term remains after we pass the output of multiplier through the LPF because the first and second terms are high-frequency.

We multiply the third term by -K and have

$$\dot{x}(t) = -Ks(t)e(t)/2 = -Ks(t)(x(t) - 2\pi f_c t)/2$$

At this point, since s(t) is LOW frequency material, we can think of $s(t)$ as constant, denoted C_s in very short period and get

$$\dot{x}(t) + (K/2)C_s x(t) = KC_s \pi f_c t)$$

For large K, homogeneous portion dies out fast and we get $x(t) \approx 2\pi f_c t$ and thus $\dot{x}(t) \approx 2\pi f_c$

This approves the answers in part b where output of VCO is $\sin(2\pi f_c t)$ and the output of PLL is $2\pi f_c$

2. **Spectral Sampling:**

(a) *(15 points)* You are given a signal

$$x(t) = \frac{\sin 2\pi t}{\pi t}$$

According to the Nyquist Theorem, what sampling rate in Hertz is necessary to ensure the signal is represented losslessly by the samples $\{x_k\}$ where $x_k = x(kT)$? What are the sample values if $x(t)$ is sampled exactly at the Nyquist rate?

SOLUTION:

Since F[x(t)] = u(f+1) - u(f-1), the maximum frequency of x(t) is 1.

According to the Nyquist Theorem, $f_s \geq 2f_max = 2$

If we sample $x(t)$ exactly at the $f_s = 2$ $(T = 1/2)$,

$$x_k = x(k/2) = \frac{2\sin \pi k}{\pi k}, k \in integer$$

$$x(kT) = \begin{cases} 2 & k = 0 \\ 0 & otherwise \end{cases}$$

(b) *(15 points)* Suppose $x(t)$ is sampled at 10Hz. Also suppose that it is known that a low pass filter with cutoff frequency ± 1Hz will be used to reconstruct $x(t)$. What number of samples *must* be sent to ensure perfect reconstruction of x(t) from the samples $\{x_k\}$? If this number is fewer than infinity, which sample(s) should be sent?

SOLUTION:

In this problem, we only have to send 1 sample (at k=0) to reconstruct x(t).

Prove mathematically:

When we pass the single sample (impulse at t=0) through a lowpass filter with cutoff frequency ± 1Hz, we multiply the spectrum of the impulse and the lowpass filter.

$$F[\delta(t)][u(f+1) - u(f-1)] = [u(f+1) - u(f-1)]$$

which is the spectrum of the x(t). The signal is reconstructed

(c) *(15 points)* Suppose we have two signals $s(t)$ and $v(t)$ both bandlimited to $\pm B$Hz and we convolve them to obtain $z(t)$. Prove that $z(t)$ is bandlimited to $\pm B$Hz.

SOLUTION:

Convolution in time domain is the multiplication in frequency domain.

$$z(t) = s(t) * v(t) => Z(f) = S(f)V(f)$$

Thus, if both $S(f)$ and $V(f)$ are bandlimited to $\pm BHz$, $Z(f)$ is also bandlimited to $\pm BHz$

(d) *(15 points)* Assume sample sets $\{s_k\}$ and $\{v_k\}$ are obtained by periodic sampling with period T where $1/T$ is the Nyquist rate. The reconstructions of $s(t)$ and $v(t)$ are

$$s(t) = \sum_k s_k g(t - kT)$$

and

$$v(t) = \sum_m v_m g(t - mT)$$

where

$$g(t) = A\frac{\sin 2\pi Bt}{\pi t}$$

with A some appropriate constant. Please derive an expression for the sequence $\{z_k\}$ in terms of the $\{s_k\}$ and $\{v_k\}$.

HINT: Take your time (or come back to this one later). Notice that $g(t) = g(-t)$ and you may assume that

$$\int_{-\infty}^{\infty} g(t - kT)g(t - mT)dt = \begin{cases} 1 & m = k \\ 0 & \text{otherwise} \end{cases}$$

This property is called signal "orthogonality" (but you don't know that yet :)).

SOLUTION:

$$\begin{aligned} z(t) &= s(t) * v(t) \\ &= \int_{-\infty}^{\infty} s(\tau)v(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \sum_k s_k g(\tau - kT) \sum_m v_m g(t - \tau - mT) \\ &= \int_{-\infty}^{\infty} \sum_k s_k g(\tau - kT) \sum_m v_m g(\tau - t + mT) \text{ //// because g(t)=g(-t)} \\ &= \int_{-\infty}^{\infty} \sum_k s_k g(\tau - kT) \sum_m v_m g(\tau - (\tfrac{t}{T} - m)T) \end{aligned}$$

From the hint, if $k = \frac{t}{T} - m$,

$z(t) = \sum_i s_i v_{\frac{t}{T}-i} = \sum_i s_i v_{\frac{t}{T}-i}\delta(t - kT)$

$z_k = \sum_i s_i v_{k-i}$

3. **Cora and the Bi-Polar Squirrel:** Cora the communications engineer has finally captured Marty the squirrel! She keeps Marty in a specially designed room whose light intensity is tailored to Marty's moods which are readily measurable via wireless electrodes attached to his feet. The problem is that if the error between the light level and Marty's mood differs by

more than ϵ, Marty goes berserk. A berserk Marty is a terror to behold and will easily bust out of the room.

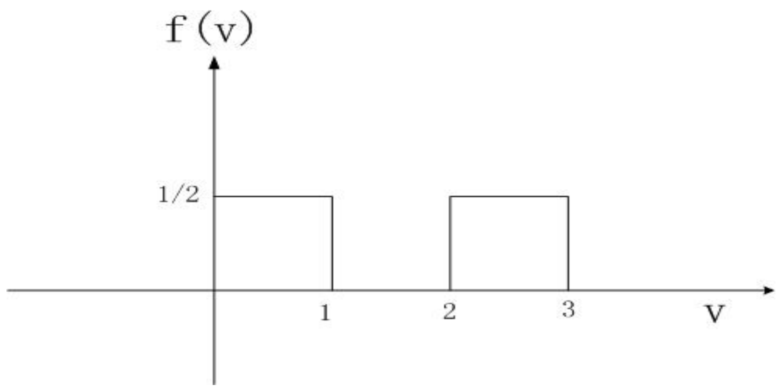
Cora carefully measures the voltages and derives the following PDF:

$$f_V(v) = \frac{u(v) - u(v-1)}{2} + \frac{u(v-2) - u(v-3)}{2}$$

Your job is to help Cora design an optimal 2-bit quantizer *according to the Lloyd-Max criteria*.

(a) *(20 points)* Please carefully sketch $f_V(v)$.

SOLUTION:



(b) *(20 points)* Determine the optimal levels $\{q_k\}$ and optimal bin boundaries $\{s_k\}$ for quantizing the voltage from Marty's electrodes. Show your results are optimal via their satisfaction of the Lloyd-Max conditions.

HINT: *You can try brute force, but making educated guesses, checking and then changing your guesses might be easier.*

SOLUTION:

Begin with guesses $s_0 = -\infty, s_1 = 1/2, s_2 = 3/2, s_3 = 5/2, s_4 = \infty$

Applying $q_k = E[S|S \in (s_k, s_{k+1})]$, *we can get corresponding quantization levels* $q_0 = 1/4, q_1 = 3/4, q_2 = 9/4, q_3 = 11/9$. *(For uniformly distributed PDF, the expected value is the midpoint in each bin)*

Then we go back to check $s_k = \frac{q_k + q_{k-1}}{2}$.

Since Lloyd-Max criteria are satisfied, the quantizer is optimal.

(c) *(20 points)* For your quantizer, what is the probability of a rampaging Marty if $\epsilon = \frac{1}{2}$?

SOLUTION:

In the proposed quantizer in 3.b, the maximum ϵ *is* $\frac{1}{4}$ *so the probability of a rampaging Marty equals zero.*