THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2004**

Quiz II

There are FOUR questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(20 points)* **Quantization I:** You are given a sequence of digits 000110111001 Suppose the sequence of digits correspond to the output of a 2-bit quantizer

$$Q(x) = \left\{ \begin{array}{ll} -1.5 & x < \frac{-1}{2} \\ -0.5 & \frac{-1}{2} \leq x < 0 \\ 0.5 & 0 \leq x < \frac{1}{2} \\ 1.5 & x \geq \frac{1}{2} \end{array} \right\}$$

with the outputs labeled '00', '01', '10', and '11' from smallest level to highest level of $Q(x)$. If samples are taken at 1 second intervals starting at $t = 0$, CAREFULLY sketch the corresponding waveform $\hat{s}(t)$ when the sequence is played (left to right) through the decoder associated with $Q(x)$.

SOLUTION: *See FIGURES 1 and 2*

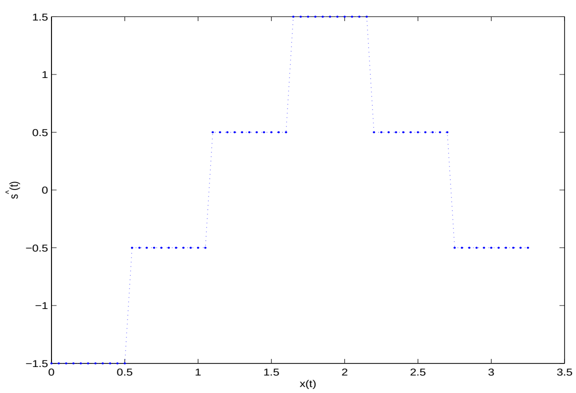

Figure 1: Problem 1 reconstructed output

2. *(30 points)* **Quantization II:** You are going to derive x_0, q_0 and q_1 for the optimum 2-level quantizer for a signal whose density function is

$$f_S(s) = \frac{1}{2}u_{-2}(s)$$

for $0 \leq t \leq 2$ where $u_{-2}()$ is the unit ramp function.

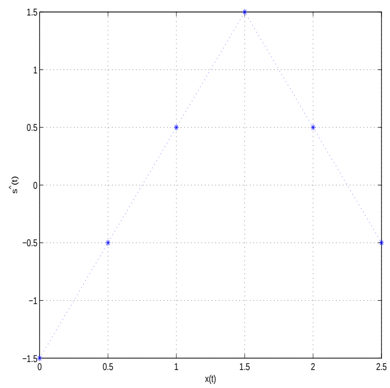

Figure 2: Problem 1, smoothed output (rough).

(a) *(10 points)* Derive an expression for q_0 solely in terms of x_0.
SOLUTION: *Lloyd-Max says:*

$$E[S|S < x_0] = q_0$$

so first we need the conditional probability that $S < x_0$

$$Prob[S < x_0] = \int_0^{x_0} \frac{1}{2} u_{-2}(s) ds = \frac{1}{4} x_0^2$$

Then we calculate

$$E[S|S < x_0] = \frac{4}{x_0^2} \int_0^{x_0} \frac{1}{2} s u_{-2}(s) ds = \frac{2}{x_0^2} \int_0^{x_0} s^2 ds = \frac{2}{x_0^2} \frac{x_0^3}{3} = \frac{2}{3} x_0$$

so,

$$q_0 = \frac{2}{3} x_0$$

(b) *(10 points)* Derive an expression for q_1 solely in terms of x_0.
SOLUTION: *Lloyd-Max says*

$$x_0 = \frac{1}{2}(q_0 + q_1)$$

so

$$q_1 = 2x_0 - q_0 = \frac{4}{3} x_0$$

(c) *(10 points)* Find the optimal value of x_0.
USEFUL FACTOID: $x^3 - 8x + 8 = (x-2)(x - (\sqrt{5} - 1))(x + (\sqrt{5} + 1))$
SOLUTION:

$$E[S|S > x_0] = q_1$$

First we calculate

$$Prob[S \geq x_0] = 1 - Prob[S < x_0]\, 1 - \frac{1}{4} x_0^2 \equiv \alpha$$

so that

$$E[S|S > x_0] = \frac{1}{\alpha}\int_{x_0}^{2} \frac{1}{2} s u_{-2}(s) ds = \frac{1}{\alpha}\frac{1}{6}(8 - x_0^3) = q_1$$

Rearranging we have

$$6\alpha q_1 = 8 - x_0^3$$

Substituting for α *we obtain*

$$(8 - 2x_0^2)x_0 = 8 - x_0^3$$

which results in

$$x_0^3 - 8x_0 + 8 = 0$$

Using the factoid we have

$$x_0 = \sqrt{5} - 1 = 1.236$$

3. *(50 points)* **Pulse Amplitude Modulation:** A signal $s(t)$ is band limited to $\pm W$ Hz and we wish to code this signal using pulse amplitude modulation. The pulse waveform is $g(t)$ and the periodic pulse train $p(t)$ by which we multiply $s(t)$ is

$$p(t) = \sum_{k=-\infty}^{\infty} g(t - kT)$$

where T is the period of the periodic pulse train So that the pulse amplitude modulated signal is

$$\tilde{s}(t) = p(t)s(t)$$

At the receiver we will apply a low pass filter to $\tilde{s}(t)$ to recover $s(t)$. In this problem you will determine when complete recovery is possible, and when it is not.

(a) *(10 points)* If the Fourier transform of the pulse $g(t)$ is $G(f)$, please determine analytically the Fourier transform $P(f)$ of the pulse train $p(t)$.

SOLUTION: *We can represent* $p(t)$ *as*

$$p(t) = \left[\sum_{k=-\infty}^{\infty} \delta(t - kT)\right] * g(t)$$

The Fourier transform of $p(t)$ *is then*

$$P(f) = \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f - k/T)\right] G(f) = \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} G(k/T)\delta(f - k/T)\right]$$

That is, $P(f)$ *is essentially the samples spectrum of* $G(f)$.

(b) *(10 points)* Given $P(f)$ and $S(f)$, please determine analytically the Fourier transform $\tilde{S}(f)$ of the signal $\tilde{s}(t)$. Expand your answer using your previous result for $P(f)$.

SOLUTION: *Since* $\tilde{s}(t)$ *is the product of two signals, its Fourier transform is the convolution of the spectra. So we have*

$$\tilde{S}(f) = \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} G(k/T)\delta(f - k/T)\right] * S(f) = \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} G(k/T)S(f - k/T)\right]$$

(c) *(10 points)* We apply a perfect low pass filter $H(f) = 1$ for $|f| \leq W$ and zero elsewhere. What condition on T must be satisfied so that we can recover $s(t)$ from $\tilde{s}(t)$? It might help to draw a picture (but only analytic answers will receive credit).

SOLUTION: *We don't want adjacent spectra $S(f \pm k/T)$ to overlap with $S(f)$. This will only be true if the space is at least $1/T = 2W$ or $T = \frac{1}{2W}$. You've just derived the Nyquist sampling theorem again.*

(d) *(10 points)* Assume the proper condition on T is satisfied. Suppose $g(t) = u(t+T/4) - 2u(t) + u(t-T/4)$ where $u(t)$ is the unit step function (sketch it). Can $s(t)$ be recovered from $\tilde{s}(t)$ using the low pass filter $H(f)$? Why/why not?

SOLUTION: *Let's take the Fourier transform of $g(t)$*

$$G(f) = \int_{-T/4}^{0} e^{-j2\pi ft} dt - \int_{0}^{T/4} e^{-j2\pi ft} dt$$

Doing the integrals we have

$$G(f) = \frac{-1}{j2\pi f}\left[1 - e^{j\pi f \frac{T}{2}}\right] - \frac{-1}{j2\pi f}\left[e^{-j\pi f \frac{T}{2}} - 1\right]$$

Rewriting

$$G(f) = \frac{1}{j2\pi f}\left[e^{j\pi f \frac{T}{2}} - 1\right] + \frac{1}{j2\pi f}\left[e^{-j\pi f \frac{T}{2}} - 1\right]$$

and simplifying yields

$$G(f) = \frac{1}{j\pi f}\left[\cos(\pi f \frac{T}{2}) - 1\right]$$

Taking the limit as $f \to 0$ we see that $G(0) = 0$ (using L'hopital's rule). So this pulse gives us NOTHING in the passband of our ideal low pass filter $H(f)$. So $\tilde{s}(t)$ cannot be recovered using this pulse shape.

(e) *(10 points)* Assume the proper condition on T is satisfied. What general conditions on $G(f)$ must be satisfied so that $s(t)$ can always be recovered from $\tilde{s}(t)$?

SOLUTION: *We simply have to have $G(0) \neq 0$. Analytically, ANYTHING else will do just fine, INCLUDING pulses $g(t)$ which might be big and fat so that they overlap in the waveform $p(t)$.*

4. *(50 points)* **Cora The Terminator:** Cora the communications engineer has been hired by SquirrelNOT, Inc. to predict the future positions of a particularly pesky squirrel named Martin P. Sciuridae. Given a sequence of his last known locations $\{s_k\}$, $k = 0, \ldots N-1$ Cora is charged to develop a linear predictor for his position at time step N, s_N. This is a one-shot deal, so Cora will either continue to be employed or will be fired based on how good her prediction is. Specifically, the company plans to use it's patented thermonuclear squirrel eradicator at the predicted location. If Martin is there, he's history, but if not, such an attempt will just make him angry. And there is nothing like an angry squirrel.

The form of her linear predictor is

$$\hat{s}_N = \sum_{k=0}^{N-1} s_k w_k$$

where the w_k are constants. You're going to help Cora find the $\{w_k\}$ given a mean square error criterion

$$\text{MSE} = E\left[\left(s_N - \sum_{k=0}^{N-1} s_k w_k\right)^2\right]$$

(a) *(10 points)* The values of s_k, $k = 0, \ldots, N$ are random variables. Define $R_{nk} = E[s_n s_k]$ and rewrite the mean square error in terms of R_{nk}.

SOLUTION: *Expanding the MSE we have*

$$MSE = E[s_N^2] + \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} w_k w_\ell E[s_k s_\ell] - 2\sum_{k=0}^{N-1} w_k E[s_k s_N]$$

which becomes

$$MSE = R_{NN} + \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} w_k w_\ell R_{k\ell} - 2\sum_{k=0}^{N-1} w_k R_{kN}$$

(b) *(10 points)* We will assume the MSE is convex in the w_k. Please provide conditions which the w_k must satisfy such that the mean square error will be minimized.

SOLUTION: *If convex, then the stationary point where the first partials are all zero is the minimum. So, at a minimum we must have*

$$\frac{\partial MSE}{\partial w_k} = 0$$

$k = 0.1, \ldots, N-1$.

(c) *(10 points)* Find explicit expressions for the w_k in terms of the R_{nk}.

SOLUTION:

$$\frac{\partial MSE}{\partial w_\ell} = 2\sum_{k=0}^{N-1} w_k R_{k\ell} - 2R_{\ell N} = 0$$

or

$$\sum_{k=0}^{N-1} w_k R_{k\ell} = R_{\ell N}$$

for each value of ℓ.

We can write down ALL the associated N equation compactly using linear algebra as

$$\mathbf{Rw} = \mathbf{y}$$

where $\mathbf{w}$ *is a vector containing the elements w_k, R is a square matrix containing elements $R_{\ell k}$ and* $\mathbf{y}$ *is a vector containing the elements $R_{\ell N}$.*

In either representation, you have a set of N equations in N unknowns and can solve for the $\{w_k\}$.

(d) *(10 points)* Incredibly enough, Cora's linear predictive method works! So, she's now told that she must use her predictor to transmit Martin's movements to a remote location. Please draw a block diagram for a linear predictive coder which users Cora's linear predictor to produce a string of bits which represent position (essentially a souped up delta modulator) . Explain how your coder works.

SOLUTION: *See FIGURE 3.*

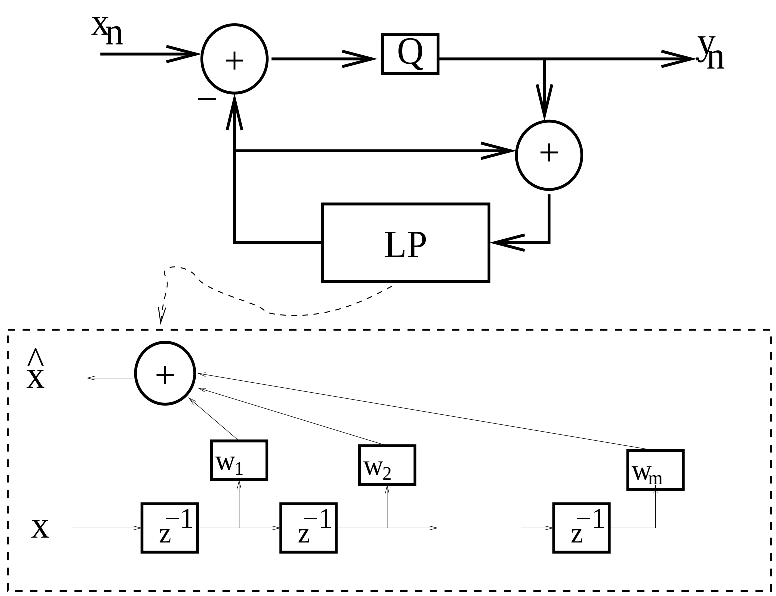

Figure 3: Cora's coder

(e) *(10 points)* Draw a block diagram of the decoder assuming Cora's linear predictive coder. How does it work?

SOLUTION: *See FIGURE 4.*

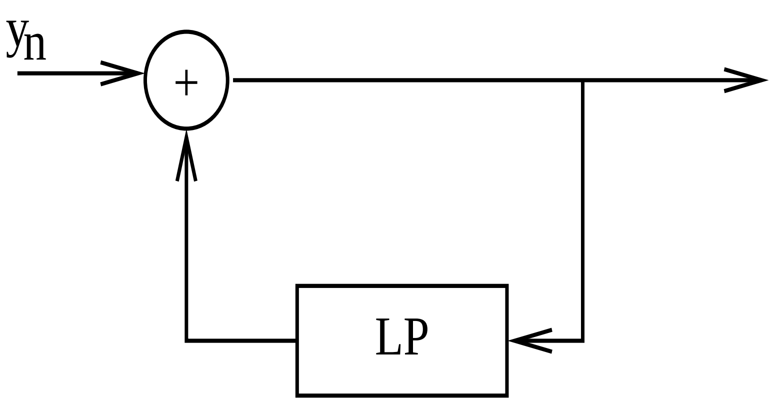

Figure 4: Cora's decoder