THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2006**

Quiz I

There are 3 questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Linear Systems Warmup:**

   (a) *(10 points)* Write down the forward and reverse Fourier Transform which relates $x(t)$ and its Fourier Transform $X(f)$.
   SOLUTION:
   $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$
   $$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

   (b) *(20 points)* Show that if $x(t)$ has Fourier Transform $X(f)$, then the Fourier Transform of $\frac{dx}{dt}$ is $j2\pi f X(f)$
   SOLUTION:
   $$\mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty}[j2\pi f X(f)]e^{j2\pi ft}df$$
   Since the last expression is a reverse FT, the function in brackets is the FT of $\frac{d}{dt}x(t)$.

   (c) *(20 points)* The energy in a signal $x(t)$ is
   $$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$
   What is the energy in the signal $x(t) = \frac{2\cos 2\pi t}{t} - \frac{\sin 2\pi t}{\pi t^2}$?
   HINT: Use your knowledge of Fourier Transforms and their properties. Also, work smart, not hard.
   SOLUTION: *We notice that*
   $$x(t) = \frac{d}{dt}\frac{\sin 2\pi t}{\pi t}$$
   and remember that via dualtiy
   $$\mathcal{F}\left\{\frac{\sin 2\pi t}{\pi t}\right\} = u(f+1) - u(f-1)$$
   so
   $$\mathcal{F}\{x(t)\} = j2\pi f(u(f+1) - u(f-1))$$

by the derivative property of the Fourier Transform. The magnitude squared is

$$(2\pi f)^2(u(f+1)-u(f-1))$$

so we can use Parseval's relation to obtain

$$\mathcal{E}_x = 4\pi^2\int_{-1}^{1} f^2 df = \frac{8\pi^2}{3}$$

2. *(50 points)* **Amplitude Modulation:** Suppose the program material is a periodic square wave

$$m(t) = \sum_k p(t-2k)$$

where $p(t) = u(t) - 2u(t-1) + u(t-2)$.

(a) *(10 points)* Sketch the AM waveform $r_1(t) = m(t)\cos 2\pi f_c t$ on the interval $[0,4]$ where $f_c = 2$

SOLUTION:

(b) *(10 points)* Sketch the large carrier AM waveform $r_2(t) = (1+m(t))\cos 2\pi f_c t$ on the interval $[0,4]$ where $f_c = 2$.

SOLUTION:

(c) *(15 points)* Now assume $f_c = 10^6$ and sketch the output waveforms when $r_1(t)$ and $r_2(t)$ are applied as inputs to ideal envelope detectors. Sketch the associated outputs.

SOLUTION: *The point is that $r_1(t)$ has constant envelope and so the output of the envelope detector is constant. $r_2(t)$ has an envelope which varies between zero and one and so $m(t)$ is recoverable by the envelope detector.*

(d) *(15 points)* Suppose $r_1(t)$ is demodulated synchronously using $\cos(2\pi f_c t + \phi)$ where ϕ is a phase offset. What is the output? You must show your work and justify your result.

SOLUTION: *The input to the low pass filter is $x(t)\cos 2\pi f_c t \cos(2\pi f_c t + \phi)$ where $x(t)$ is either $m(t)$ or $1+m(t)$. We rewrite this as*

$$x(t)\cos\phi\cos^2 2\pi f_c t - x(t)\frac{\sin\phi}{2}\sin 4\pi f_c t$$

We can expand $\cos^2$ to obtain

$$x(t)\frac{\cos\phi}{2} + x(t)\frac{\cos\phi}{2}\cos 4\pi f_c t - x(t)\frac{\sin\phi}{2}\sin 4\pi f_c t$$

Spectrally, the first term is

$$0.5\cos\phi X(f)$$

The second term is

$$0.25\cos\phi\,[X(f+2f_c) - X(f-2f_c)]$$

The third term is

$$0.5j\sin\phi[X(f+2f_c) - X(f-2f_c)]$$

Only the term $0.5\cos\phi X(f)$ will survive the LPF so we're left with $0.5\cos\phi x(t)$ as the time domain output.

3. *(50 points)* **Frequency/Phase Modulation:**

(a) *(15 points)* Frequency modulation of a signal $m(t)$ has the form

$$r(t) = \cos\left(2\pi f_c t + \beta \int_0^t m(\tau)d\tau\right)$$

where β is a constant.

Assume $f_c = 10^8$Hz, $\beta = 100$ and a single-sided bandwidth of $f_m = 10^4$Hz for the program material $m(t)$. If $r(t)$ is applied to an ideal envelop detector. What is the resulting output? How might you answer change if $f_m = 10^6$Hz?

SOLUTION: *The envelope of the signal is constant and equal to* 1*, so the envelope detector will output* 1*. However, if* β *is large enough, the bandwidth of the FM signal, approximated by Carson's rule as* $2f_m(\beta+1)$*, will be wide and potentially have components all the way to DC – where the envelope detector might pick them up because the signal might have a slowly varying component.*

(b) *(15 points)* We wish to design a superheterodyne receiver for radio stations whose carriers are between 1GHz and 2GHz and adjacent channels are separated by 200KHz. We can design tunable filters in the GHz range with Qs of only about 10. This means that our filters will have a bandwith about 10% as wide as the carrier. So for a 1GHz carrier, we can have a 100MHz wide tunable bandpass filter which cannot separate out a single 200KHz channel in the passband.

What is the smallest value intermediate frequency (IF) can we use for our heterodyne receiver?

SOLUTION: *If the IF is* f_I*, we want to make sure that the image frequency band centered around* $f_c - 2f_I$ *is outside the band of our passband filter centered around the carrier. So we must have the lower band edge of the filter* $f_c - 0.05f_c$ *greater than* $f_c - 2f_I + 100$*KHz.*

$$f_c - 0.05f_c > f_c - 2f_I + 10^5$$

or $f_I > 0.025f_c + 5\times 10^4$*. For* f_c *in the GHz range,* $0.025f_c$ *dwarfs* 10^5 *so we can ignore it. Thus we have* $f_I > 0.025f_c$ *and taking* f_c *at its maximum value of* 2*GHz we have* $f_I > 50$*MHz.*

NOTE: You'll notice that this IF is high enough and the signal band narrow enough that it still might be hard to pull out a 200KHz channel (need a filter Q of around 250*). So, we could take the heterodyned signal and apply yet another heterodyning operation to it to make the IF low enough that we can build reasonably inexpensive filters to extract the specific channel of interest.*

(c) *(20 points)* Cora the communications engineer knows that single sideband AM can be used to reduce the bandwidth of the radiated signal $r(t)$ and thereby allow radio channels to be more densely packed in frequency domain.

A phase modulated (PM) signal has the form

$$r(t) = \cos(2\pi f_c t + \beta m(t))$$

Please help Cora devise, if possible, a comparable scheme for narrowband and wide-band PM. Quantitatively argue your case. If it is possible, show how to do single sideband PM (modulation and demodulation). If not, show why not. You can assume any ideal component you'd like.

SOLUTION: *For narrowband PM ($\beta m(t) \gg 1$), the radiated signal is*

$$r(t) = \cos 2\pi f_c t - \beta m(t) \sin 2\pi f_c t$$

Since the spectrum is

$$R(f) = \frac{1}{2}(\delta(f+f_c) + \delta(f-f_c)) - \frac{j\beta}{2}(M(f+f_c) - M(f-f_c))$$

we could easily do vestigial filtering and remove the lower or upper signal sidebands. Synchronous demodulation using $\sin 2\pi f_c t$ would then recover the signal and ignore the $\cos 2\pi f_c t$ component.

For wideband PM ($\beta m(t) \gg 1$), the radiated signal is

$$r(t) = \cos(2\pi f_c t + \beta m(t)) = \cos(\beta m(t)) \cos 2\pi f_c t - \sin(\beta m(t)) \sin 2\pi f_c t$$

The key issue in SSB is whether the spectrum to the right of the carrier frequency f_c contains the same information as the spectrum to the left of the carrier frequency. Well, defining new "program material" functions

$$\phi(t) = \cos(\beta m(t))$$

and

$$\psi(t) = \sin(\beta m(t))$$

we have

$$r(t) = \phi(t) \cos 2\pi f_c t - \psi(t) \sin 2\pi f_c t$$

and spectrally we have

$$R(f) = \frac{1}{2}\left[\Phi(f+f_c) + \Phi(f-f_c)\right] - \frac{j}{2}\left[\Psi(f+f_c) - \Psi(f-f_c)\right]$$

Since the baseband spectra $\Psi(f)$ and $\Phi(f)$ must have conjugate symmetry (they are the fourier transforms of real signals), the spectrum $R(f)$ will have conjugate symmetry about the carrier frequency and once again, this will allow us do to SSB.

We can synchronously demodulate in $\sin$ and $\cos$ to obtain $\phi(t)$ and $\psi(t)$ respectively. And then, as one possible approach, we could internally reproduce the wideband $r(t)$ signal within the receiver and then use standard FM demodulation on it.

Of course, all this assumes β is not so large as to have the spectra $\Phi(f+f_c)$ and $\Phi(f-f_c)$ overlap (ditto for $\Psi()$).

So Cora can indeed do SSB FM.