THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2005**

Quiz I

There are four questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Linear Systems Quickie Melange:**

   (a) *(10 points)* Provide Fourier SERIES expansions for $\cos(2\pi t)$, $\sin(2\pi t)$ and $\cos(2\pi t) + \cos(6\pi t)$.

   SOLUTION: $f(t) = \sum_k a_k e^{j2\pi f_0 kt}$ *and*

$$a_k = \begin{cases} 1/2 & |k| = 1 \\ 0 & o.w. \end{cases}$$

$$a_k = \begin{cases} -j/2 & k = 1 \\ j/2 & k = -1 \\ 0 & o.w. \end{cases}$$

$$a_k = \begin{cases} 1/2 & |k| = 1 \\ 1/2 & |k| = 3 \\ 0 & o.w. \end{cases}$$

   (b) *(10 points)* Provide Fourier TRANSFORMS for $\cos(2\pi t)$, $\sin(2\pi t)$ and $\cos(2\pi t) + \cos(6\pi t)$.

   SOLUTION:

$$\frac{1}{2}\left(\delta(f-1) + \delta(f+1)\right)$$

$$\frac{-j}{2}\left(\delta(f-1) - \delta(f+1)\right)$$

$$\frac{1}{2}\left(\delta(f-1) + \delta(f+1)\right) + \frac{1}{2}\left(\delta(f-3) + \delta(f+3)\right)$$

   (c) *(10 points)* Provide the Fourier TRANSFORM of $\cos^2(2\pi t)$.

   SOLUTION: *Product in time domain is a convolution so*

$$\frac{1}{2}\delta(f) + \frac{1}{4}\left(\delta(f-2) + \delta(f+2)\right)$$

   Can also get the same result by using the identity $\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$ *and taking the FT directly.*

(d) *(10 points)* Find the energy in $x(t) = \frac{\sin 2\pi t}{\pi t}$.

SOLUTION: *Time domain is useless here so move to frequency domain and use Parseval. $X(f) = u(f+1) - u(f-1)$. $|X(f)|^2 = X(f)$. The area under $|X(f)|^2$ is 2 so the energy is 2.*

(e) *(10 points)* $x(t) = \frac{\sin 2\pi t}{\pi t}$ is applied to a filter with impulse response $h(t) = \frac{\sin 2\pi t}{\pi t}$. What is the output $y(t) = (x * h)(t)$?

SOLUTION: *Given the previous part we're simply multiplying the spectra $X(f)$ and $H(f)$ which are identical and of height one. So the output spectrum is the same $Y(f) = X(f) = H(f) = u(f+1) - u(f-1)$. So the output is $y(t) = x(t) = h(t)$*

2. *(50 points)* **Amplitude Modulation:** You are given two signals, $m_1(t)$ and $m_2(t)$ with spectra $M_1(f)$ and $M_2(f)$ as shown in FIGURE 1. For all parts of this problem you can assume you

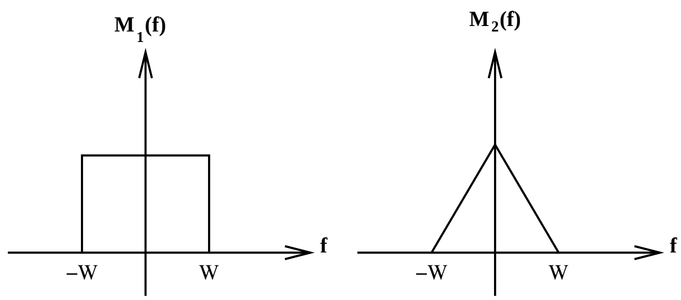

Figure 1: Spectra for problem 2.

have a modulation/demodulation toolkit which contains as many multipliers as you'd like and as many linear time invariant filters as you'd like (all ideal). You must also assume that the signals $m_1(t)$ and $m_2(t)$ must be sent simultaneously.

(a) *(15 points)* You are given oscillators which output $\cos(2\pi f_c t)$ and $\cos(4\pi f_c t)$ where $f_c \gg W$. You may assume you have access to the same oscillators at the receiver, that the phases of the oscillators at the receiver and transmitter match, and that there is no propagation-induced phase shift of the transmitted carrier at the receiver.

Please draw carefully labeled block diagrams of an AM transmitter and an associated AM receiver which provide (possibly scaled) copies of $m_1(t)$ and $m_2(t)$ at the receiver output.

SOLUTION: *Standard synchronous AM: Put $m_1(t)$ on the f_c carrier and synchronously demodulate it at the receiver. Put $m_2(t)$ on the $2_f c$ carrier. The spectra won't overlap since $W \ll f_c$. See FIGURE 2*

(b) *(15 points)* Now assume you are given oscillators which output $\cos(2\pi f_c t)$ and $\cos(4\pi f_c t)$ where $f_c \gg W$ but that no copies of the oscillators exist at the receiver. For this part you may assume you have access to ideal diodes.

Please draw carefully labeled block diagrams of an AM transmitter and an associated AM receiver which provides (possibly scaled) copies of $m_1(t)$ and $m_2(t)$ at the receiver output.

SOLUTION: *Standard large carrier AM. Filter the signals in the passband and then use envelope detectors to extract $m_1(t)$ and $m_2(t)$. See FIGURE 3*

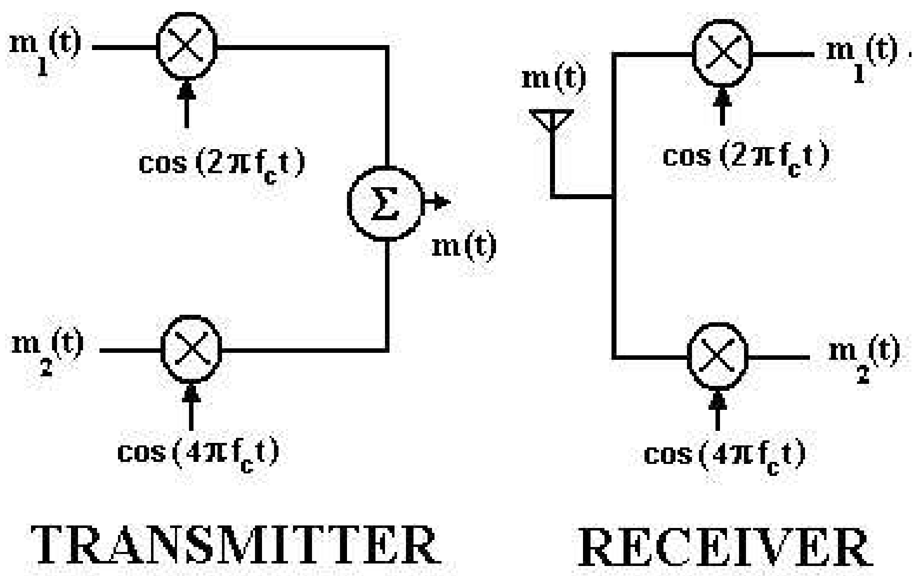

Figure 2: Sketch for Question 2a

(c) *(20 points)* Now assume you are given oscillators which output $\cos(2\pi f_c t)$ and $\cos(2\pi(f_c + W)t)$ where $f_c \gg W$, but otherwise assume the conditions of part 2a.

Please draw carefully labeled block diagrams of an AM transmitter and an associated AM receiver which provides (possibly scaled) copies of $m_1(t)$ and $m_2(t)$ at the receiver output.

SOLUTION: *Two ways to go: could do single sideband using Hilbert or passband filters, or could multiply carriers to obtain a second carrier at double the frequency (and DC, but can filter that out before doing the modulation and demodulation). See FIGURES 4, 5,6*

3. *(50 points)* **Cora's Boxed PLL:** Cora the Communications Engineer has been charged with setting up a phase locked loop for her employer. She has taken the PCS course a number of times and knows all about phase locked loops using sinusoids. However, her employer, Boxomatic, is in the middle of its "Box the world!" campaign and tells Cora to use only "boxy" signals. Thus, the input to the phase locked loop (see FIGURE 7) is $c(2\pi f_c t) = \text{sgn}(\cos 2\pi f_c t)$ and the voltage-controlled oscillator output is $s(\phi(t))$ where $s(t) = \text{sgn}(\sin t)$.

   Please show that in the limit of large Γ as given in FIGURE 7 that $\phi(t) \approx 2\pi f_c t$. State all assumptions and approximations.

   SOLUTION: *The key to this problem is figuring out the output of that low pass filter so we can close the loop and derive some differential equations just like we did in class for the "classical" PLL. So, $c(2\pi f_c t)$ and $s(2\pi f_c t)$ are periodic square waves with period $T = 1/f_c$ and offset by a quarter cycle. Sketching out $c(2\pi f_c t)$ and $s(2\pi f_c t)$ and sliding them past*

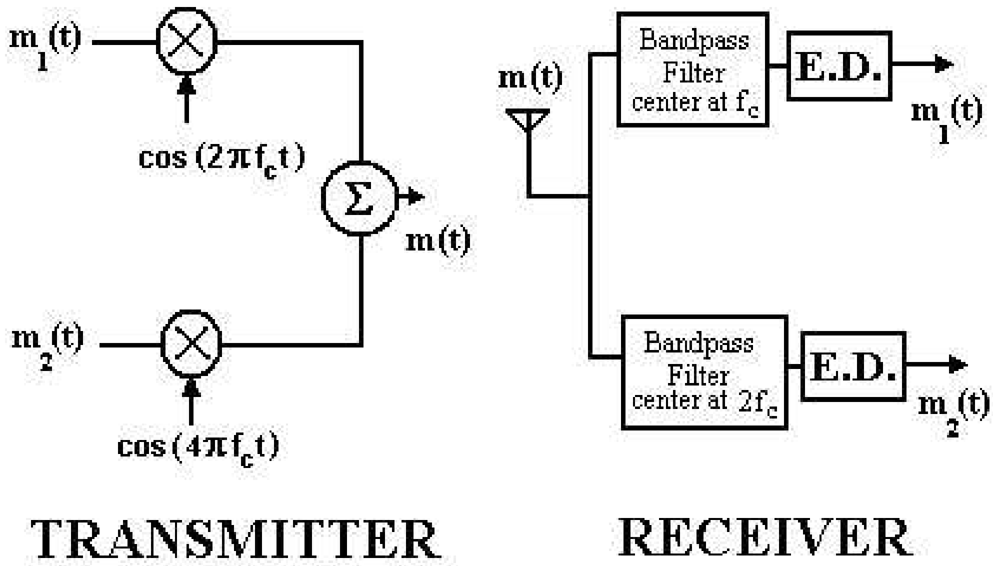

Figure 3: Sketch for Question 2b

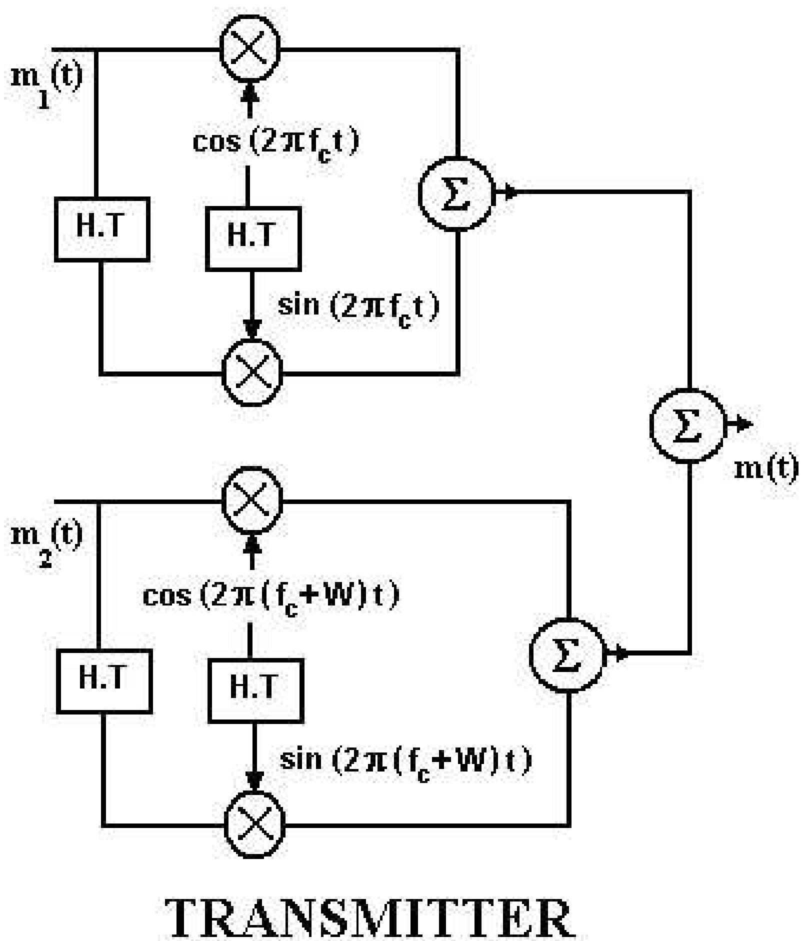

Figure 4: Sketch for a possible transmitter (using SSB, lower side band) for Question 2c

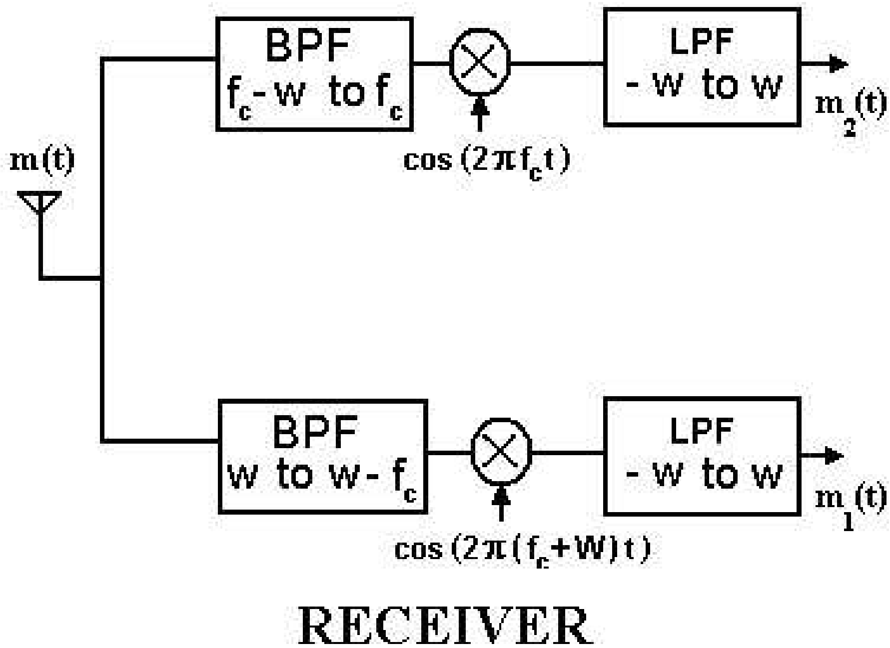

Figure 5: Sketch for the corresponding receier (using SSB lower side band) for Question 2c

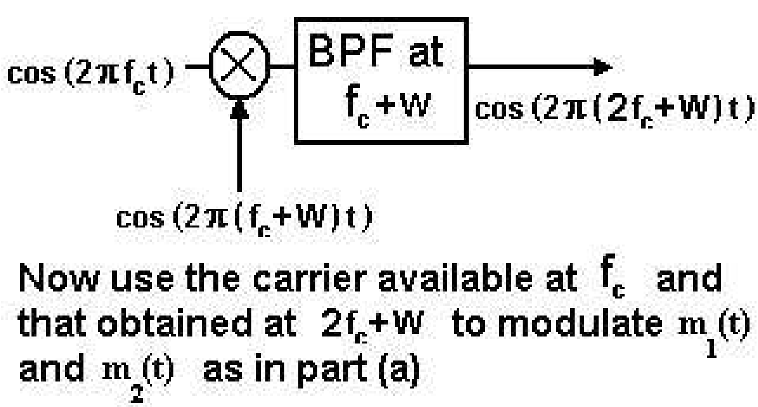

Figure 6: Another way to do question 2c

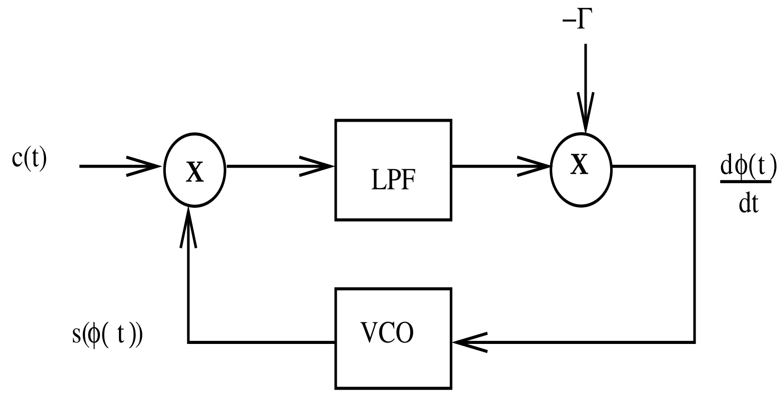

Figure 7: Phase locked loop diagram for problem 3.

each other reveals that the average $A(\tau)$ of $c(2\pi f_c t)s(2\pi f_c(t-\tau))$ is periodic and equal to

$$A(\tau) = \begin{cases} \dfrac{4(\tau+\frac{T}{2})}{T} & \frac{-T}{2} \le t \le \frac{-T}{4} \\ \dfrac{-4\tau}{T} & \frac{-T}{4} \le t \le \frac{T}{4} \\ \dfrac{4(\tau-\frac{T}{2})}{T} & \frac{T}{4} \le t \le \frac{T}{2} \end{cases}$$

The part we're going to end up being interested in is the central part which is linear with negative slope about $\tau = 0$.

So, the output of the multiplier is $c(2\pi f_c t)s(\phi(t))$ which we rewrite as $c(2\pi f_c t)s(2\pi f_c(t-\epsilon(t)))$ where we'll assume epsilon is small, just as in class. Using our result for $A(\tau)$, the output of the filter is then $-4\epsilon(t)/T$ and we have

$$\frac{d\phi(t)}{dt} = 4\Gamma\epsilon(t)$$

Substituting for $\phi(t) = 2\pi f_c(t-\epsilon(t))$ yields

$$\frac{d\phi(t)}{dt} = -4\Gamma(\phi(t)/(2\pi f_c) - t)$$

which we rewrite as

$$2\pi f_c \frac{d\phi(t)}{dt} + 4\Gamma\phi(t) = 4\Gamma(2\pi f_c t)$$

The homogeneous solution of this constant coefficient linear differential equation decays since the root is $-2\Gamma/(\pi f_c)$ and if we make Γ large we have approximately

$$+4\Gamma\phi(t) \approx 4\Gamma(2\pi f_c t)$$

or

$$\phi(t) \approx 2\pi f_c t$$

So this loop using these boxy signals works too! In fact, the VCO for a REAL phase locked loop DOES output a square wave so you've now analyzed the case of the practical PLL! Don't you feel like a master of the universe?

4. *(50 points)* **Frequency/Phase Modulation:**

(a) *(25 points)* Let $r(t) = A\cos(2\pi f_c t + \beta m(t))$ where A is a constant. For $|\beta| \ll 1$ provide an approximate analytic expression for the spectrum of $r(t)$.

SOLUTION: *Expand cosine:*

$$r(t) = A\left(\cos(2\pi f_c t)\cos(\beta m(t)) - \sin(2\pi f_c t)\sin(\beta m(t))\right) \approx A\left(\cos(2\pi f_c t) - \beta m(t)\sin(2\pi f_c t)\right)$$

The approximate spectrum is then

$$R(f) \approx \frac{A}{2}\left(\delta(f+f_c) + \delta(f-f_c)\right) + \frac{-Aj}{2}\left(M(f-f_c) - M(f+f_c)\right)$$

(b) *(25 points)* Let $r(t) = A\cos(2\pi f_c t + \beta f(t))$ where A is a constant. PROVE that the average power in the signal $r(t)$ is $A^2/2$ for periodic $f(t)$ independent of the value of β. You may assume that the period T of $f(t)$ obeys $f_c \gg 1/T$.

SOLUTION: *The average power is the integral of $r^2(t)$ over a period. We have*

$$r^2(t) = \frac{A^2}{2}\left[1 + \cos(4\pi f_c t + 2\beta f(t))\right]$$

Now, for the astute among you, you recognized that that's as far as you could go given the problem information. That is, suppose $f(t) = -2\pi f_c t/\beta$ (or at least a suitable periodic wrap-around of that function). However, if you in addition assume $|2\beta f(t)| \ll 2\pi$ it's immediately apparent that you have a high speed sinusoid in $\cos(4\pi f_c t + 2\beta f(t))$ which integrates to zero and you're left with the constant $A^2/2$.

Following through under the assumptions (or making additional assumptions) gave you full credit. If you got courageous and tried time warping or some other such, you get a smiley face (and full credit if it worked out).