THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2003**

Quiz I

There are three questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **F-T Properties for Comm. Theorists:**

   (a) *(10 points)* The Fourier transform of $x(t)$ is $X(f)$. PROVE that the Fourier transform of $x(t-t_0)$ is $e^{-j2\pi f t_0}X(f)$.

   SOLUTION:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi ft}dt &= \int_{-\infty-t_0}^{\infty-t_0} x(z)e^{-j2\pi f(z+t_0)}dz \\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} x(z)e^{-j2\pi fz}dz \\ &= e^{-j2\pi ft_0}X(f)\end{aligned}$$

   (b) *(10 points)* The Fourier transform of $x(t)$ is $X(f)$. PROVE that the Fourier transform of $\frac{dx(t)}{dt}$ is $j2\pi fX(f)$.

   SOLUTION:

$$\begin{aligned}\frac{dx(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df &= \int_{-\infty}^{\infty} X(f)\frac{d}{dt}\left(e^{j2\pi ft}\right)df \\ &= \int_{-\infty}^{\infty} j2\pi fX(f)e^{j2\pi ft}df \\ &= \int_{-\infty}^{\infty} X'(f).df\end{aligned}$$

   where $X'(f) = j2\pi fX(f)$ *is the Fourier transform of* $\frac{dx(t)}{dt}$.

   (c) *(10 points)* The Fourier transform of $x(t)$ is $X(f)$. Show that the Fourier transform of $x(t)\cos(2\pi f_c t)$ is $\frac{1}{2}(X(f-f_c)+X(f+f_c))$

   SOLUTION: *The Fourier transform of* $\cos 2\pi f_c t$ *is* $\frac{1}{2}(\delta(f+f_c)+\delta(f-f_c))$. *We can verify this by taking the inverse transform and using the sifting property of impulses to get* $\frac{1}{2}(e^{-j2\pi f_c t}+e^{j2\pi f_c t}) = \cos 2\pi f_c t$ *Then since multiplication in time domain means convolution in frequency domain we have*

$$\frac{1}{2}(\delta(f+f_c)+\delta(f-f_c)) * X(f) = \frac{1}{2}(X(f+f_c)+X(f-f_c))$$

   which is the desired result.

   (d) *(20 points)* Let $X(f) = u(f+f_c+1)-u(f+f_c-1)+u(f-f_c+1)-u(f-f_c-1)$. We form $y(t) = x(t)\cos(2\pi f_c t)$ and then apply a filter with impulse response $h(t) = \frac{\sin 2\pi t}{\pi t}$ to obtain $z(t) = y(t) * h(t)$. Given $f_c \gg 1$, what is $Z(f)$?

SOLUTION: *We just showed in the previous part what multiplying by cosine does. So we have* $Y(f) = \frac{1}{2}(X(f-f_c)+X(f+f_c))$*. So all that's left is to determine the transform of* $h(t)$*. Well, we know that a rectangular pulse in time is a sinc in frequency. By duality we know that a rectangular pulse in frequency is thus a sinc in time – which is what we have. However, I always forget the scale factors, so to avoid problems I almost always just assume a general rectangle and do the inverse transform.*

$$\int_{-A}^{A} Qe^{j2\pi ft}df = Q\frac{\sin(2\pi At)}{\pi t}$$

Equating the constants gives $Q = A = 1$*. So now we know that* $H(f) = u(f+1) - u(f-1)$*.*

To finish up, we calculate

$$\begin{aligned}\tfrac{1}{2}(X(f-f_c)+X(f+f_c)) &= (u(f+1)-u(f-1)) \\ &+ \tfrac{1}{2}(u(f+f_c+1)-u(f+f_c-1)) \\ &+ \tfrac{1}{2}(u(f-f_c+1)-u(f-f_c-1))\end{aligned}$$

which when multiplied by $H(f)$ *gives*

$$Z(f) = (u(f+1)-u(f-1))$$

By the way – you just did your first AM demodulation!

2. *(50 points)* **Fun With Gaussians:** X and Y are both zero mean, unit variance jointly Gaussian random variables with correlation coefficient ρ. Let $Z_1 = a_1X + b_1Y$ and $Z_2 = a_2X + b_2Y$ where the $\{a_i\}$ and $\{b_i\}$ are arbitary constants. Assume that Z_1 and Z_2 are always jointly Gaussian, and find an expression which the $\{a_i\}$ and $\{b_i\}$ must satisfy so that Z_1 and Z_2 are independent. Do NOT solve for the $\{a_i\}$ and $\{b_i\}$ explicitly.

   SOLUTION: *Because* Z_1 *and* Z_2 *are jointly Gaussian, if the correlation coefficient between them is zero, they must be independent. Since they are zero mean,* $\rho_{Z_1Z_2} = 0$ *implies* $E[Z_1Z_2] = 0$*. Now,*

$$Z_1Z_2 = a_1a_2X^2 + b_1b_2Y^2 + (a_1b_2 + a_2b_1)XY$$

   which implies

$$E[Z_1Z_2] = a_1a_2\sigma_X^2 + b_1b_2\sigma_Y^2 + (a_1b_2 + a_2b_1)\rho\sigma_X\sigma_Y = 0$$

   which is good enough.

   EXTRA INFO: *For those of you who have had linear algebra, you'll recognize that the above is a statement that*

$$\mathbf{a}^\top \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_X \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix} \mathbf{b} = 0$$

   where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

   and

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

You'll recognize the matrix as the covariance matrix for X and Y The statement (independence of Z_1 and Z_2) will always be true if **a** *and* **b** *are orthogonal eigenvectors of the covariance matrix (and such eigenvectors can always be found for a symmetric matrix).*

3. *(50 points)* **Cora and the E-Bug:** Mhos-R-Us, Inc. makes very accurate resistors for the aerospace industry and has noticed small unpredictable fluctuations across their resistors. When probes are attached, a stochastic waveform $X(t)$ is observed. Cora the Communications Engineer has been hired to figure out what is wrong with Mhos-R-Us resistors and since Cora is a Rutgers E&CE alumna, you've been asked to help her.

   At first, Cora is stumped and almost ready to give up when a coworker accidentally applies high voltage to the probes she's holding and knocks her out cold. While she's out, Cora dreams about a little bug who jumps back and forth across the terminals of a teensy resistor. She realizes these bugs are individual electrons and when they jump between the teensy terminals, ohm's law must be obeyed and a small voltage pulse is produced. When the bug moves from, say, the left to right terminals at time $t = t_0$, a voltage pulse

$$p(t) = ae^{-\kappa(t-t_0)}u(t-t_0)$$

   where a and κ are constants. Right to left jumping produces $-p(t)$. Meanwhile, Cora's coworker tries to help by splashing her with a bucket of cold water. But since he's not removed the probes from her hands or turned off the voltage, the effect is to drive Cora deeper into her revelatory dream.

   Cora suddenly realizes that this particular form of pulse is what you get out of a capacitor in series with a resistor when you apply an impulse. She realizes that the sudden bug motion must produce a voltage spike (impulse) and that even the teensiest resistor has SOME capacitance! She models the voltage applied by the jumping process as

$$J(t) = \sum_k (-1)^k \delta(t-t_k)$$

   where t_k is the time of the k^{th} jump. Each jump elicits a response $h(t-t_k) = ae^{-\kappa(t-t_k)}$ so that the voltage across the resistor is

$$X(t) = \sum_k (-1)^k p(t-t_k) = J(t) * h(t)$$

   Cora then realizes that the bug jumps randomly but in a statistically predictable way – the time between jumps is an exponentially distributed random variable Δ where $f_\Delta(d) = \lambda e^{-\lambda d}$ and each interval is independent of all other intervals. The voltage waveform is therefore a stochastic process.

   Cora then imagines a whole army (N) of teensy resistors lined up end to end with independently jumping bugs and realizes that the voltage across the resulting resistor must be

$$V(t) = \sum_{n=1}^{N} X_n(t)$$

   where $X_n(t)$ is the voltage across the n^{th} resistor. Her coworker turns off the voltage, Cora wakes up, relates her dream to you and faints dead away. It's up to you to finish the job.

(a) *(10 points)* A random telegraph wave stochastic process $W(t)$ takes on values ± 1 with independent, identically distributed intervals between transitions. If the intervals between transitions are each distributed as $\lambda e^{-\lambda t}$, then the autocorrelation function is $R_W(\tau) = e^{-2\lambda|\tau|}$. Carefully sketch a possible sample function of this process. Do the same for $J(t)$.

SOLUTION: *We've sketched $W(t)$ in class. What you'll notice is that differentiating $W(t)$ gives you something which looks exactly like a possible $J(t)$ – a train of alternating impulses. So you can think of $J(t)$ as the process which results when you pass $W(t)$ through a filter $j2\pi f$ (see the first problem).*

(b) *(10 points)* What is $S_J(f)$, the power spectral density of $J(t)$?

SOLUTION: *Since $J(t)$ is the derivative of $W(t)$ we have $S_J(f) = (2\pi f)^2 S_W(f)$ We find*

$$S_W(f) = \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-j2\pi f\tau} d\tau = \int_0^{\infty} e^{-2\lambda\tau - j2\pi f\tau} d\tau + \int_{-\infty}^{0} e^{2\lambda\tau - j2\pi f\tau} d\tau$$

so that

$$S_W(f) = \frac{1}{2\lambda + j2\pi f} + \frac{1}{2\lambda - j2\pi f} = \frac{4\lambda}{4\lambda^2 + (2\pi f)^2}$$

Thus,

$$S_J(f) = \frac{4\lambda(2\pi f)^2}{4\lambda^2 + (2\pi f)^2}$$

(c) *(10 points)* What is $S_X(f)$, the power spectral density of $X(t)$ in terms of $S_J(f)$?

SOLUTION: *We just pass $J(t)$ through the filter $h(t)$. First*

$$H(f) = \int_0^{\infty} a e^{-\kappa t} e^{-j2\pi f t} dt = \frac{a}{\kappa + j2\pi f}$$

so that

$$|H(f)|^2 = \frac{a^2}{\kappa^2 + (2\pi f)^2}$$

and thus

$$S_X(f) = |H(f)|^2 S_J(f) = \frac{a^2}{\kappa^2 + (2\pi f)^2} S_J(f)$$

(d) *(10 points)* What is $S_V(f)$, the power spectral density of $V(t)$ in terms of $S_X(f)$?

SOLUTION: *First off, $J(t)$ is derived from LTI filtering of a zero mean stationary process, so $J(t)$ is zero mean. Likewise, $X(t)$ is zero mean. So, suppose you have a bunch of independent zero mean processes $X_n(t)$. We then have $E[X_k(t)X_m(t+\tau)] = 0$ unless $m = k$ so that $E[(\sum_n X_n(t))(\sum_m X_m(t+\tau))] = \sum_{n=1}^{N} R_X(\tau) = NR_X(\tau)$.*

So in total we have

$$S_V(f) = NS_X(f)$$

(e) *(10 points)* Find an expression for the variance of $V(t)$. For N large, find an approximate probability density function for $V(t)$.

SOLUTION:

$$\sigma^2_{V(t)} = E[V^2(t)] = R_V(0) = \int_{-\infty}^{\infty} S_V(f)df$$

What we have at a given time t is a sum of IID random variables which makes us think of the central limit theorem and the Gaussian approximation. All we need is the mean and the variance. The mean we've already established as zero. So the appoximate density is

$$f_{V(t)}(v) = \frac{1}{\sqrt{2\pi\sigma^2_{V(t)}}} e^{-v^2/2\sigma^2_{V(t)}}$$

(f) *Bonus Question (10 points)* Evaluate $R_V(0)$ explicitly.

SOLUTION: *From answers to previous parts,*

$$R_V(0) = \int_{-\infty}^{\infty} S_V(f)df = Na^2 4\lambda 4\pi^2 \int_{-\infty}^{\infty} \frac{f^2}{(\kappa^2 + 4\pi^2 f^2)(4\lambda^2 + 4\pi^2 f^2)}$$

Expanding in partial fractions,

$$\begin{aligned} R_v(0) &= Na^2 4\lambda 4\pi^2 [\tfrac{\kappa^2}{4\pi^2(\kappa^2 - 4\lambda^2)} \textstyle\int_{-\infty}^{\infty} \tfrac{1}{(\kappa^2 + 4\pi^2 f^2)}\, df \\ &\quad + \tfrac{\lambda^2}{\pi^2(4\lambda^2 - \kappa^2)} \textstyle\int_{-\infty}^{\infty} \tfrac{1}{4\lambda^2 + 4\pi^2 f^2}\, df] \\ &= Na^2 4\lambda 4\pi^2 [\tfrac{\kappa^2}{(4\pi^2)^2(\kappa^2 - 4\lambda^2)} \textstyle\int_{-\infty}^{\infty} \tfrac{1}{f^2 + (\frac{\kappa}{2\pi})^2}\, df \\ &\quad + \tfrac{\lambda^2}{4\pi^2\pi^2(\kappa^2 - 4\lambda^2)} \textstyle\int_{-\infty}^{\infty} \tfrac{1}{f^2 + (\frac{\lambda}{\pi})^2}\, df] \end{aligned}$$

Applying the formula,

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}(\frac{x}{a})$$

and the result that at $x = \infty$

$$\tan^{-1}(\frac{x}{a}) = \frac{a}{|a|} \frac{\pi}{2}$$

we get the final result,

$$R_v(0) = \frac{2Na^2\lambda}{(4\lambda^2 - \kappa^2)} [2|\lambda| - |\kappa|]$$