THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2004**

Problem Set 9

Haykin: 1.1–1.10

1. Consider a random process $X(t)$ defined by

$$X(t) = sin(2\pi f_c t)$$

in which the frequency f_c is a random variable uniformly distributed over the range $[0, W]$. Show that $X(t)$ is nonstationary. Hint: Examine specific sample functions of the random process $X(t)$ for the frequency $f = W/2$, $W/4$ and W say.

SOLUTION: *An easy way to solve this problem is to find the mean of the random process* $X(t)$

$$E[X(t)] = \frac{1}{W}\int_0^W \sin(2\pi f t)\, df = \frac{1}{W}[1 - \cos(2\pi W t)]$$

Clearly $E[X(t)]$ *is a function of time and hence the process* $X(t)$ *is not stationary.*

2. Let X and Y be statistically independent Gaussian-distributed random variables each with zero mean and unit variance. Define the Gaussian process

$$Z(t) = X\ cos(2\pi t) + Y\ sin(2\pi t)$$

(a) Determine the joint probability density function of the random variables $Z(t_1)$ and $Z(t_2)$ obtained by observing $Z(t)$ at times t_1 and t_2 respectively.

SOLUTION: *Since every weighted sum of the samples of the Gaussian process* $Z(t)$ *is Gaussian,* $Z(t_1)$, $Z(t_2)$ *are jointly Gaussian random variables. Hence we need to find mean, variance and correlation co-efficient to evaluate the joint Gaussian PDF.*

$$E[Z(t_1)] = cos(2\pi t_1)E[X] + sin(2\pi t_1)E[Y]$$

Since $E[X] = E[Y] = 0$, $E[Z(t_1)] = 0$. *Similarly,* $E[Z(t_2)] = 0$.

$$\begin{aligned}
Cov[Z(t_1)Z(t_2)] &= E[Z(t_1)Z(t_2)] \\
&= E[X\cos(2\pi t_1) + Y\sin(2\pi t_1)][X\cos(2\pi t_2) + Y\sin(2\pi t_2)] \\
&= \cos(2\pi t_1)\cos(2\pi t_2)E[X^2] + [\cos(2\pi t_1)sin(2\pi t_2) + \sin(2\pi t_1)\cos(2\pi t_2)]E[XY] \\
&\quad + \sin(2\pi t_1)\sin(2\pi t_2)E[Y^2]
\end{aligned}$$

Noting that, $E[X^2] = 1$, $E[Y^2] = 1$ *and* $E[XY] = E[X].E[Y] = 0$ *(since* X *and* Y *are independent), we obtain,*

$$Cov[Z(t_1)Z(t_2)] = cos[2\pi(t_1 - t_2)]$$

$\sigma^2_{Z(t_1)} = E[Z^2(t_1)] = 1$. *This result is obtained by putting* $t_1 = t_2$ *in* $Cov[Z(t_1)Z(t_2)]$.

Similarly, $\sigma^2_{Z(t_2)} = E[Z^2(t_2)] = 1$

Correlation coefficient is given by

$$\rho = \frac{Cov[Z(t_1)Z(t_2)]}{\sigma_{Z(t_1)}\sigma^2_{Z(t_2)}} = \cos[2\pi(t_1 - t_2)]$$

Hence the joint PDF

$$f_{Z(t_1),Z(t_2)}(z_1, z_2) = C.exp[-Q(z_1, z_2)]$$

where,

$$C = \frac{1}{2\pi\sqrt{(1-\cos^2(2\pi(t_1-t_2)))}} = \frac{1}{2\pi\sin[2\pi(t_1-t_2)]}$$

$$Q(z_1, z_2) = \frac{1}{\sin^2[2\pi(t_1-t_2)]}[z_1^2 - 2\cos[2\pi(t_1-t_2)]z_1z_2 + z_2^2]$$

(b) Is the process $Z(t)$ stationary? Why?
SOLUTION: *We find that* $E[Z(t)] = 0$ *and covariance of* $Z(t_1)$ *and* $Z(t_2)$ *depends only on the time difference* $t_1 - t_2$. *The process* $Z(t)$ *is hence wide sense stationary. Since it is Gaussian, it is also strict sense stationary.*

3. The square wave $x(t)$ of FIGURE 1 of constant amplitude A, period T_0, and delay t_d, represents the sample function of a random process $X(t)$. The delay is random, described by the probability density function

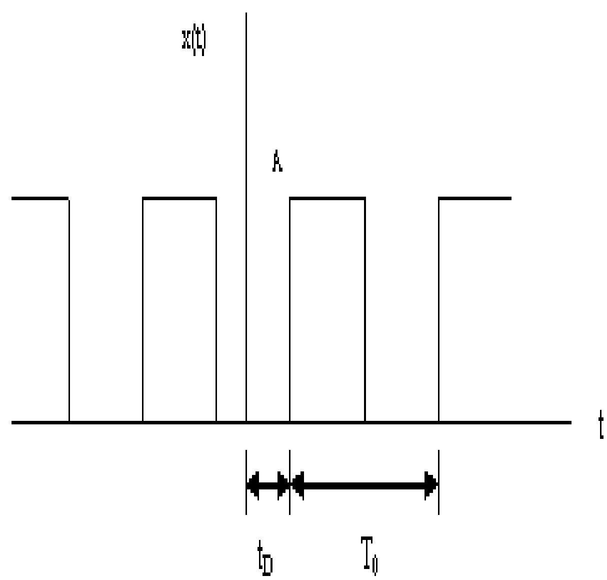

Figure 1: Square wave for $x(t)$

$$f_{T_D}(t_d) = \begin{cases} \frac{1}{T_0} & \frac{-T_0}{2} \leq t_d \leq \frac{T_0}{2} \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the probability density function of the random variable $X(t_k)$ obtained by observing the random process $X(t)$ at time t_k.

SOLUTION: *$X(t)$ is a square wave, and it takes on the two values 0 or A with equal probability. Hence the PDF can be given as*

$$f_{X(t)}(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x - A)$$

(b) Determine the mean and autocorrelation function of $X(t)$ using ensemble-averaging

SOLUTION: *Using our definition of ensemble average for the mean of a stochastic process, we find*

$$\begin{aligned} E[X(t)] &= \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx \\ &= 0.\tfrac{1}{2} + A.\tfrac{1}{2} \\ &= \tfrac{A}{2} \end{aligned}$$

Autocorrelation: Let's denote the square wave with random delay time t_D, period T_0 and amplitude A as $A.Sq_{T_0}(t - t_D)$. Then, the autocorrelation can be written as,

$$\begin{aligned} R_{X(\tau)} &= E[A.Sq_{T_0}(t - t_D).A.Sq_{T_0}(t - t_D + \tau)] \\ &= A^2 \int_{-\infty}^{\infty} Sq_{T_0}(t - t_D).Sq_{T_0}(t - t_D + \tau)]\, f_{T_D}(t_D)\, dt_D \\ &= A^2 \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} Sq_{T_0}(t - t_D).Sq_{T_0}(t - t_D + \tau)]\tfrac{1}{T_0}\, dt_D \\ &= \tfrac{A^2}{2}(1 - 2\tfrac{|\tau|}{T_0}), |\tau| \le \tfrac{T_0}{2} \end{aligned}$$

Since the square wave is periodic with period T_0, $R_X(t)$ must also be periodic with period T_0.

(c) Determine the mean and autocorrelation function of $X(t)$ using time-averaging.

SOLUTION: *On a time-averaging basis we note by inspection that the mean is*

$$< x(t) >= \frac{A}{2}$$

and time-autocorrelation is,

$$< x(t+\tau)x(t) >= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t+\tau)x(t) = \frac{A^2}{2}(1 - 2\frac{|\tau|}{T_0}), |\tau| \le \frac{T_0}{2}$$

Again, the autocorrelation is periodic with period T_0.

(d) Establish whether or not $X(t)$ is stationary. In what sense is it ergodic?

SOLUTION: *We note that the ensemble-averaging and time-averaging yield the same set of results for the mean and autocorrelation functions. Therefore, $X(t)$ is ergodic in the mean and autocorrelation function. Since ergodicity implies wide-sense stationarity, it follows that $X(t)$ must be wide-sense stationary.*

4. Consider two linear filters connected in cascade as in FIGURE 2. Let $X(t)$ be a stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and second filter output is $Y(t)$.

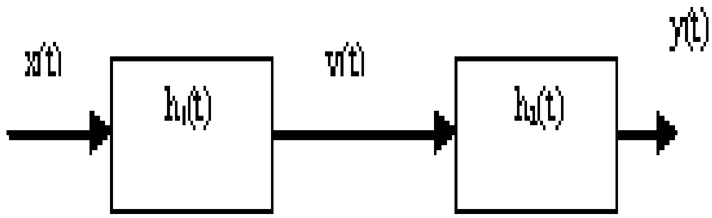

Figure 2: Cascade of linear filters

(a) Find the autocorrelation function of $Y(t)$
SOLUTION: *The cascade connection of two filters is equivalent to a filter with impulse response*

$$h(t) = \int_{-\infty}^{\infty} h_1(u) h_2(t-u)\, du$$

The autocorrelation function of $Y(t)$ is given by,

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

(b) Find the cross-correlation function $R_{VY}(\tau)$ of $V(t)$ and $Y(t)$.
SOLUTION: *The cross correlation function of $V(t)$ and $Y(t)$ is,*

$$R_{VY}(\tau) = E[V(t+\tau)Y(t)]$$

$V(t)$ and $Y(t)$ are related as follows,

$$Y(t) = \int_{-\infty}^{\infty} V(\lambda)h_2(t-\lambda)\, d\lambda$$

Therefore,

$$\begin{aligned} R_{VY}(\tau) &= E[V(t+\tau)\textstyle\int_{-\infty}^{\infty} V(\lambda)h_2(t-\lambda)\, d\lambda] \\ &= \textstyle\int_{-\infty}^{\infty} h_2(t-\lambda)E[V(t+\tau)V(\lambda)]\, d\lambda \\ &= \textstyle\int_{-\infty}^{\infty} h_2(t-\lambda)R_V(t+\tau-\lambda)\, d\lambda \end{aligned}$$

5. A random telegraph signal $X(t)$, characterized by the autocorrelation function

$$R_X(\tau) = \exp(-2v|\tau|)$$

where v is a constant, is applied to a low-pass RC filter of FIGURE 3. Determine the power spectral density and autocorrelation function of the random process at the filter output.

SOLUTION: *The power spectral density of the random telegraph wave is given as,*

$$\begin{aligned} S_X(f) &= \textstyle\int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi ft)\, dt \\ &= \textstyle\int_{-\infty}^{0} \exp(2vt)\exp(-j2\pi ft)\, dt + \int_0^{\infty}\exp(-2vt)\exp(-j2\pi ft)\, dt \\ &= \tfrac{1}{2(v-j\pi f)} + \tfrac{1}{2(v+j\pi f)} \\ &= \tfrac{v}{v^2+\pi^2 f^2} \end{aligned}$$

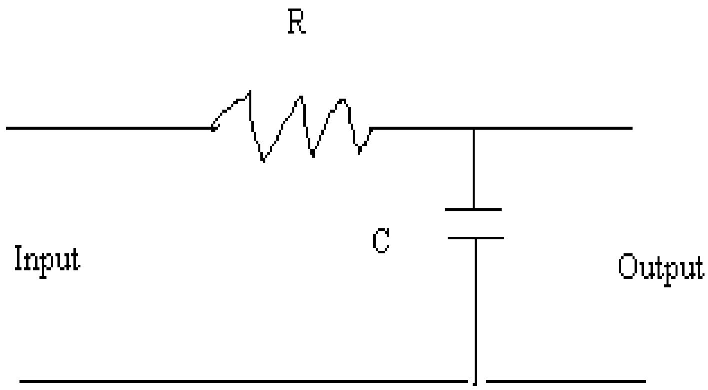

Figure 3: RC Filter

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Therefore, PSD of the filter output is,

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{v}{(v^2 + \pi^2 f^2)[1 + j2\pi fRC]}$$

To find the autocorrelation function of the filter output, we first expand $S_Y(f)$ in partial fractions as follows,

$$S_Y(f) = \frac{v}{1 - 4R^2C^2v^2}[\frac{-1}{(1/2RC)^2 + \pi^2 f^2} + \frac{1}{v^2 + \pi^2 f^2}]$$

Recognizing that,

$$IFT\{\frac{1}{v^2 + \pi^2 f^2}\} = \exp(-2v|\tau|)$$

$$IFT\{[\frac{(1/2RC)}{(1/2RC)^2 + \pi^2 f^2}\} = exp(-|t|/RC)$$

where IFT stands for Inverse Fourier Transform, we get $R_Y(\tau) = IFT\{S_Y(f)\}$

$$R_Y(\tau) = \frac{v}{1 - 4R^2C^2v^2}[\frac{-\exp(-2v|\tau|)}{v} - 2RC\exp(-\frac{|\tau|}{RC})]$$

6. A stationary Gaussian process $X(t)$ has zero mean and power spectral density $S_X(f)$. Determine the probability density function of a random variable obtained by observing the process $X(t)$ at some time t_k.

   SOLUTION: *Let $\mu(x)$ be the mean and $\sigma^2(x)$ be the variance of the random variable X_k obtained by observing the random process at time t_k. Then,*

$$\mu_x = 0$$

$$\sigma_x^2 = E[X_k^2] - \mu_x^2 = E[X_k^2]$$

We note that

$$\sigma_x^2 = E[X_k^2] = \int_{-\infty}^{\infty} S_X(f)\, df$$

The PDF of Gaussian random variable X_k is given by

$$f_{X_k}(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp(\frac{-x^2}{2\sigma_x^2})$$

7. A stationary Gaussian process $X(t)$ with mean μ_x and variance σ_X^2 is passed through two linear filters with impulse responses $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$, as shown in FIGURE 4

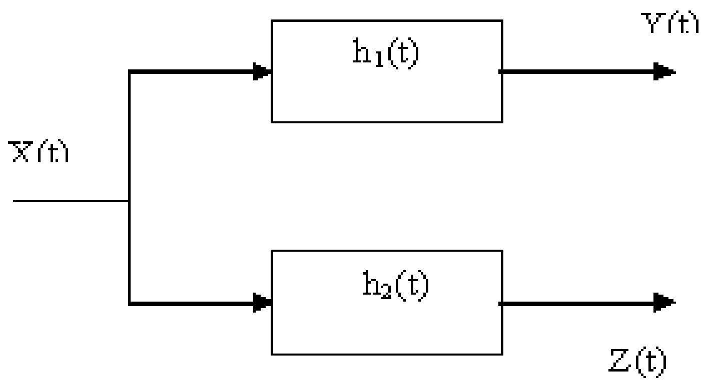

Figure 4: Parallel systems.

(a) Determine the joint probability density function of the random variables $Y(t_1)$ and $Z(t_2)$.

SOLUTION: *Since $X(t)$ is a Gaussian random process, the random variables $Y(t_1)$ and $Z(t_2)$ are jointly Gaussian. Hence to find the joint PDF, we need to find variance $\sigma_{Y_1}^2$, $\sigma_{Z_2}^2$, mean μ_{Y_1}, μ_{Z_2} and correlation coefficient* $\rho = \frac{cov[Y(t_1)Z(t_2)]}{\sigma_{Y_1}\sigma_{Z_2}}$

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1 - \tau) h_1(\tau)\, d\tau$$

$$\mu_{Y_1} = H_1(0)\mu(x)$$

where $H_1(0) = \int_{-\infty}^{\infty} h_1(\tau)\, d\tau$ and μ_x is the mean of the stationary random process $X(t)$. Similarly,

$$Z(t_2) = \int_{-\infty}^{\infty} X(t_2 - u) h_2(u)\, du$$

$$\mu_{Z_2} = H_2(0)\mu(x)$$

where $H_2(0) = \int_{-\infty}^{\infty} h_2(u)\, du$ and μ_x is the mean of the stationary random process $X(t)$.

The covariance of $Y(t_1)$ *and* $Z(t_2)$ *is*

$$\begin{aligned}
Cov[Y(t_1)Z(t_2)] &= E[(Y(t_1)-\mu_{Y_1})(Z(t_2)-\mu_{Z_2})] \\
&= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X(t_1-\tau)-\mu_x)(X(t_2)-\mu_x)h_1(\tau)h_2(u)\,d\tau\,du] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E[(X(t_1-\tau)-\mu_x)(X(t_2)-\mu_x)]h_1(\tau)h_2(u)\,d\tau\,du \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_X(t_1-t_2-\tau+u)h_1(\tau)h_2(u)\,d\tau\,du
\end{aligned}$$

where $C_X(\tau)$ *is the autocovariance function of* $X(t)$*.*

$$\begin{aligned}
\sigma_{Y_1}^2 &= E[(Y(t_1)-\mu_{Y_1})^2] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_X(\tau-u)h_1(\tau)h_1(u)\,d\tau\,du \\
\sigma_{Z_2}^2 &= E[(Z(t_2)-\mu_{Z_2})^2] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_X(\tau-u)h_2(\tau)h_2(u)\,d\tau\,du
\end{aligned}$$

Finally, the joint Gaussian PDF can be written as,

$$f_{Y(t_1),Z(t_2)}(y,z) = k\exp -Q(y,z)$$

where,

$$k = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Z_2}\sqrt{1-\rho^2}}$$

$$Q(y,z) = \frac{1}{2(1-\rho^2)}[(\frac{y-\mu_{Y_1}}{\sigma_{Y_1}})^2 - 2\rho(\frac{y-\mu_{Y_1}}{\sigma_{Y_1}})(\frac{z-\mu_{Z_2}}{\sigma_{Z_2}}) + (\frac{z-\mu_{Z_2}}{\sigma_{Z_2}})^2]$$

(b) What conditions are necessary and sufficient to ensure that $Y(t_1)$ and $Z(t_2)$ are statistically independent?

SOLUTION: *The random variables* $Y(t_1)$ *and* $Z(t_2)$ *are uncorrelated if and only if their covariance is zero. Since* $Y(t)$ *and* $Z(t)$ *are jointly Gaussian processes, it follows that* $Y(t_1)$ *and* $Z(t_2)$ *are statistically independent if* $Cov[Y(t_1)Z(t_2)] = 0$*. Therefore the necessary and sufficient condition for* $Y(t_1)$ *and* $Z(t_2)$ *to be statistically independent is that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_X(t_1-t_2-\tau+u)h_1(\tau)h_2(u)\,d\tau\,du$$

8. A stationary Gaussian process $X(t)$ with zero mean and power spectral density $S_X(f)$ is applied to a linear filter whose impulse response $h(t)$ is shown in FIGURE 5. A sample Y is

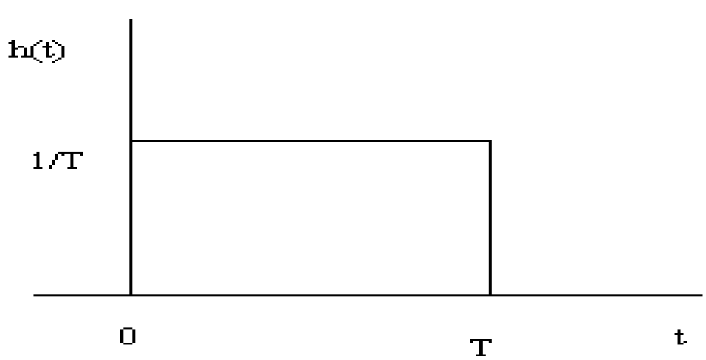

Figure 5: $h(t)$ for problem 8

taken of the random process at the filter output at time T.

(a) Determine the mean and variance of Y

SOLUTION: *The filter output is*

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau \\ &= \frac{1}{T}\int_0^T X(T-\tau)\, d\tau \end{aligned}$$

Put $T-\tau = u$. Then the sample value of $Y(t)$ at $t = T$ equals

$$Y = \frac{1}{T}\int_0^T X(u)\, du$$

The mean of Y is therefore

$$\begin{aligned} E[Y] &= E[\tfrac{1}{T}\int_0^T X(u)\, du] \\ &= \tfrac{1}{T}\int_0^T E[X(u)]\, du \\ &= 0 \end{aligned}$$

Variance of Y

$$\begin{aligned} \sigma_Y^2 &= E[Y^2]-E[Y]^2 \\ &= R_Y(0) \\ &= \int_{-\infty}^{\infty} S_Y(f)\, df \\ &= \int_{-\infty}^{\infty} S_X(f)|H(f)|^2\, df \\ H(f) &= \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)\, dt \\ &= \tfrac{1}{T}\int_0^T \exp(-j2\pi ft)\, dt \\ &= sinc(fT)\exp(-j\pi fT) \end{aligned}$$

Therefore,

$$\sigma_Y^2 = \int_{-\infty}^{\infty} S_X(f) sinc^2(fT)\, df$$

(b) What is the probability density function of Y?

SOLUTION: *Since the filter output is Gaussian, it follows that Y is also Gaussian. Hence the PDF of Y is*

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}}\exp(\frac{-y^2}{2\sigma_Y^2})$$